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References to reviews in Mathematical Reviews before volume 20 (1959) are by volume and page number, as MR 19, 532: from volume 20 on, by volume and review number, as MR **20** #4387. Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

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Mathematical Reviews

Vol 22, No. 4A | April, 1961 | Reviews 2510–3020

GENERAL

2510:

Rothstein, Jerome. Physical demonology. Methodos **11** (1959), 99–121.

E. T. Whittaker [*From Euclid to Eddington*, Cambridge Univ. Press, New York, 1949; MR **11**, 2; p. 59] suggested that much, if not all, of physics could be stated in the form of "postulates of impotence". The author, although he overlooks this reference, argues the case for it in a very interesting and amusing manner. He defines a demon who is potent where we are impotent, and he describes a physical law as "anti-demon legislation". A law states that "no physicist, no matter how fiendishly clever..., can hope to be a demon." The paper makes more obvious "the organic connection between the growth of science and the cultural milieu." There is much emphasis on the relevance of information theory to science.

The author emphasizes that a quantitative measure for the amount of information contained in a law can be defined provided that the experimental conditions are given. This idea may be compared with I. J. Good, Proc. Inst. Elec. Engrs. C. (3) **103** (1956), 200–204 [MR **17**, 868], Section 5. *I. J. Good* (Teddington)

2511:

Stebbing, L. Susan. ★Philosophy and the physicists. Dover Publications, Inc., New York, 1958. xvii+295 pp. Paperbound: $1.65.

Susan Stebbing's work formed an essential part of the training of the student of philosophy and logic during the 2nd quarter of this century and her *Modern introduction to logic* [Methuen, London, 1930] is still a classic on the subject. In 1937 she brought out *Philosophy and the physicists*, which may well rank as one of the best introductions to the range of problems it discusses, such as the logical status of physical explanations; the question of the analyticity of some of the major principles of science (e.g., laws of motion; law of gravitation; conservation and relativity principles); the nature of the mechanistic world-picture of the 19th century scientist when contrasted with that of the next, with its rejection of physical determinism and its doctrine of the ultimacy of the statistical laws of quantum mechanics, including a searching discussion of the consequences of this approach to the question of human free-will.

But the book is more than a perfunctory introduction to these problems. It is, to echo the title of another of Miss Stebbing's books, an introduction to "clear thinking". It presents the revolt of the philosopher against the often incredibly muddled thinking of even the greatest of our 20th century physicists, committing mistakes which —as this volume shows so well—were made and corrected hundreds of years ago, in the hands of Locke, Berkeley, Hume, Kant and Mill, to mention just some of the relevant names. We can understand Miss Stebbing's indignant horror at the spectacle; the text abounds in expressions such as "misleading nonsense" (p. 84), "perverse argument" (93), "inextricably confused" (98), "muddle" (125), "absurd mistake" (183), "entirely unwarranted" (199). Sometimes perhaps she does less than justice to the attempts of Jeans and Eddington to render in poetic language what a concise mathematical argument would fail to convey to the layman. But in the later portions of her book, her criticisms hit home. She is not the only one to have applied a searching critique to Eddington's view of the circularity of physics; witness for instance the later attempts of R. B. Braithwaite and Max Born. But here the criticism is pushed with a qualitative care which may give even the non-physicist a chance to appraise the value of the argument. There are interesting if doubtful applications of what has since come to be called the method of "paradigm-case-argument", as when Eddington's joke that a plank on which we stand is not solid but like a swarm of flies through which we may slip perchance, is encountered by the objection (p. 52) that "there is no common usage of language that provides a meaning for the word 'solid' that would make sense to say that the plank on which I stand is not solid." She makes the important point that physical theories explain the sequence of our sense-data by means of concepts, only in the sense that they provide correlations between the two groups (p. 63). And the whole dramatic story that leads from Locke, via Berkeley and Hume to Kant, is here once more played out, with the Eddington-Locke doctrine of the unknowable external object as the hidden and at best inferred cause of our sensations (e.g., p. 128), matched by the Kant-Stebbing reply that "it is only because [we] *first* know that there are external objects that [we] are able to *infer* that there are private sensa." (p. 130).

The picture of outworn metaphysics combined with a popularised version of the individual scientist's interpretation of recently discovered physical phenomena produces an inextricable hotchpotch of stimulating obiter dicta together with a great deal of downright error. Perhaps Professor Stebbing's has so far only been a still small voice. But I cannot believe that physics and the modern worldview based on it will not ultimately benefit by drawing on centuries of accumulated attempts at clarification of simple and fundamental issues which the academic philosophers have made since the time of Plato and Aristotle. What she implies is that the philosopher has the right—on his own ground—to expect an observance of the proprieties of scholarship which the working physicist is only too eager to claim on behalf of his own work. *G. Buchdahl* (Cambridge, England)

2512:

Bouligand, G. Effort théorique et limites au pouvoir d'unifier. Dialectica 14 (1960), 167–175.

2513:

★**Manual of the dozen system.** Duodecimal Society of America, Inc., New York, 1960. v + 33 pp. $1.00.

A presentation in popular style of the argument for 12 as a base, including elementary problems, brief tables of logarithms, sines, weights and measures, etc.

2514:

Burington, Richard Stevens. ★Handbook of mathematical tables and formulas. 3rd ed. Handbook Publishers, Inc., Sandusky, Ohio, 1949 (reprinted: 1958). viii + 296 pp. $3.50.

Thirty standard mathematical tables and seven sections of formulas chiefly from the calculus and vector analysis.

2515:

★**Труды третьего всесоюзного математического съезда, Москва, июнь-июль 1956. Том IV. [Proceedings of the third all-Union mathematical conference, Moscow, June-July, 1956. Vol. IV.]** Izdat. Akad. Nauk SSSR, Moscow, 1959. 250 pp. 15 r.

Preceding volumes were listed in MR **20** #6973. Vol. IV consists of summaries of sectional reports and of reports by foreign scientists.

2516:

Bokštein, M. F. The Second All-Union Topological Conference. Uspehi Mat. Nauk **15** (1960), no. 3 (93), 203–224. (Russian)

HISTORY AND BIOGRAPHY

See also B3533.

2517:

Lorenzen, Paul. ★Die Entstehung der exakten Wissenschaften. Verständliche Wissenschaft, Bd. 72. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960. v + 163 pp. DM 8.80.

After some introductory remarks on the value of the history of science and the meaning of the term "exact sciences" a concise summary of pre-Hellenic science (Egyptian and Babylonian mathematics and astronomy) is given. The author then discusses in full the science of the Greeks (geometry, arithmetic, theory of music, astronomy, trigonometry, logic, mechanics, physics). He then deals with the origins of modern science (physics and mathematics in the Middle Ages, the astronomy of Copernicus and Kepler, the contributions of Chr. Huygens to dynamics, the fundamental work of Isaac Newton, the development of mathematics in the 16th–18th centuries, the growth of formal logic and the theory of probability in the 19th and 20th centuries). All these subjects are treated very concisely but in perfect clearness.

E. J. Dijksterhuis (Bilthoven)

2518:

Réalisations des sciences mathématiques en Roumanie au cours des quinze dernières années. Rev. Math. Pures Appl. **4** (1959), 337–340.

2519:

L'œuvre accomplie dans le domaine des sciences mathématiques, dans la République Populaire Roumaine, au cours des quinze dernières années. Acad. R. P. Romîne. Stud. Cerc. Mat. **10** (1959), 243–245. (Romanian)

2520:

Fifteen years' development of applications of mathematics in the Czechoslovak republic (1945–1960). Apl. Mat. **5** (1960), 159–169. (Czech)

2521:

Lohne, Johs. Hooke versus Newton. An analysis of the documents in the case on free fall and planetary motion. Centaurus **7** (1960), 6–52.

2522:

Lohne, Johs. Thomas Harriott (1560–1621). The Tycho Brahe of optics. Centaurus **6** (1959), 113–121. (5 plates)

2523:

Sekerž-Zen'kovič, Ya. I. Aleksandr Ivanovič Nekrasov (on the 75th anniversary of his birth). Uspehi Mat. Nauk **15** (1960), no. 1 (91), 153–162. (1 plate) (Russian)

A short general and scientific biography with a photograph and a bibliography of 68 entries.

2524:

Aleksandrov, P. S.; Višik, M. I.; Saul'ev, V. K.; Èl'sgol'c, L. È. Lazar' Aronovič Lyusternik (on the sixtieth anniversary of his birth). Uspehi Mat. Nauk **15** (1960), no. 2 (92), 215–230. (1 plate) (Russian)

A short general and scientific biography with a photograph and a bibliography of 125 entries.

2525:

Kreĭn, M. G.; Šilov, G. E. Mark Aronovič Naĭmark (on the fiftieth anniversary of his birth). Uspehi Mat. Nauk **15** (1960), no. 2 (92), 231–236. (1 plate) (Russian)

A short general and scientific biography with a photograph and a bibliography of 72 entries.

2526:

Gagnebin, S. Gonseth tel que je le connais. Dialectica **14** (1960), 109–120.

2527:

Publications de Ferdinand Gonseth. Dialectica **14** (1960), 267–274.

2528:
Hornich, Hans. Johann Radon. Jber. Deutsch. Math. Verein. **63**, Abt. 1, 51–52 (1960).

For a fuller biography, the author refers to P. Funk, Monatsh. Math. **62** (1958), 189–199 [MR **20** #2254].

2529:
Löbell, Frank. Obituary: Konrad Knopp. Jbuch. Bayer. Akad. Wiss. München **1958**, 187–189.

2530:
Fenchel, Werner. Jakob Nielsen in memoriam. Acta Math. **103** (1960), vii–xix.

2531:
Hj. Tallqvist (Obituary notice). Arkhimedes **1958**, no. 2, 52–54. (Swedish)

2532:
Scorza, Gaetano. ★Opere scelte. Vol. I (1899–1915). Pubblicate a cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1960. 513 pp. (1 plate) L. 5000.

2533:
Eulerus, Leonhardus. ★Opera omnia. Series secunda. Opera mechanica et astronomica. Vol. XXVIII. Commentationes astronomicae ad theoriam motuum planetarum et cometarum pertinentes. Edidit Leo Courvoisier. Societas Scientiarum Naturalium Helveticae, Zurich, 1959. xl+332 pp.

The present volume of Euler's collected works contains the following seven articles: De motu planetarum et orbitarum determinatione; Orbitae solaris determinatio; Solutio problematum quorundam astronomicorum; Determinatio orbitae cometae qui mense Martio huius anni 1742 potissimum fuit observatus; Theoria motuum planetarum et cometarum; Mémoire sur la plus grande équation des planètes; Recherches et calculs sur la vraie orbite elliptique de la comète de l'an 1769 et son temps pèriodique. At the end of the volume appears an article, "Recherches sur le dérangement d'une comète qui passe près d'une planète" by Nicolaus Fuss, who was only 18 years old at the time the article was written. Fuss himself mentions that Euler deserves credit for any merit which the article may have, and it is apparently for this reason the article is included in Euler's works.

In the introduction the editor, Leo Courvoisier, deals separately with each article. His careful analysis follows the original text very closely, and the reading of the Latin text has thus been facilitated greatly. The editor has made no attempt, however, to place the articles in a proper historical frame. References to earlier works on similar subjects are missing as are also any mentioning of the possible survival of the methods used by Euler. These objections are of minor importance. Leo Courvoisier has apparently preferred to use his energy on the purely editorial work which also includes the checking of all numerical computations. The following laconical remark with which the editor concludes his analysis of each article gives a vivid impression of the spirit with which he carried out his task. He writes: "Der Unterzeichnete hat das Werk, wie immer, in mühsamer Kleinarbeit bis ins Einzelne revidiert und korrigiert. Von den Verbesserungen der zahlreichen Fehler in den numerischen Rechnungen und einigen Formeln sind die wichtigsten durch Fussnoten angezeigt worden."
O. Schmidt (Copenhagen)

LOGIC AND FOUNDATIONS

2534:
Aebi, M. Die Stellung von Gonseths "offener Philosophie" im ganzen der Philosophia perennis. Dialectica **14** (1960), 127–150.

2535:
Bernays, Paul. Charakterzüge der Philosophie Gonseths. Dialectica **14** (1960), 151–156.

2536:
Quine, Willard Van Orman. ★Word and object. Studies in Communication. The Technology Press, Cambridge, Mass.; John Wiley & Sons, Inc., New York-London; 1960. xv+294 pp. $5.50.

This book is concerned primarily with natural language and only very derivatively with logico-mathematical systems. Nor is the author in any way concerned with the use of mathematical technics in the study of natural language such as those used in structural linguistics. His interest here focusses rather upon how the meaning and reference of certain kinds of words are learned. This is discussed primarily in terms of dispositions to respond to socially observable stimuli. Problems concerned with translation, synonymy, analyticity, existence, abstract objects, time, modality, and nominalism are touched upon inter alia.

The author's philosophy of logic is clearer here than in any of his previous writings. "On the whole," he says, "the canonical systems of logical notation are best seen not as complete notations for discourse on special subjects, but as partial notations for discourse on all subjects." A mathematician may wish to use quantifiers over just the natural numbers, for example, but it is "best" for him to think of them as covering all objects whatsoever. The meaning of "best" here requires a more thorough analysis and defence than the author gives it.
R. M. Martin (Bonn)

2537:
v. Freytag-Löringhoff, Bruno. ★Logik: Ihr System und ihr Verhältnis zur Logistik. 2. verbesserte Aufl. Urban-Bücher, Bd. 16. W. Kohlhammer Verlag, Stuttgart, 1955. 224 pp. DM 3.60.

Der grösste Teil des oben genannten Buches behandelt den Aufbau eines Logiksystems, das für philosophische Untersuchungen brauchbar sein mag; für die Grundlagenforschung der Mathematik ist dieses System jedenfalls ungeeignet, weil der Aufbau vor allem in den Grundbegriffen zu unpräzise ist.

Im zweiten Teil des Buches geht der Verfasser auch auf die mathematische Logik ein. Die Darstellung dieser Disziplin ist jedoch nicht frei von schwerwiegenden Fehlern. Ein Beispiel möge das belegen. Auf Seite 167 schreibt der Verfasser: "Sie [die Implikation] hat im Aussagenkalkül und darüber hinaus in der gesamten Logistik, so weit sie sich auf ihn stützt, die Aufgabe, das logische Folgern darzustellen." Angesichts solcher Missverständnisse sind dann auch bei der folgenden Diskussion über das Verhältnis der Logistik zur Logik keine interessanten Ergebnisse zu erwarten.

H. Kiesow (Oberhausen)

2538:

Stegmüller, Wolfgang. ★Unvollständigkeit und Unentscheidbarkeit. Die metamathematischen Resultate von Gödel, Church, Kleene, Rosser und ihre erkenntnistheoretische Bedeutung. Springer-Verlag, Vienna, 1959. iii+114 pp. $4.70.

This book consists of a welcome exposition of recent metamathematical developments. But the mathematical reader will find here nothing new, nor will the philosophical reader find even a cursory discussion of the epistemological significance of metamathematics. The subtitle is misleading. There is no mention of any epistemological matter except in the Preface and there only in passing.

R. M. Martin (Bonn)

2539:

Myhill, John. Some remarks on the notion of proof. J. Philos. **57** (1960), 461–471.

This paper deals with some logico-philosophical comments concerning the feasibility of methods of 'proof' which are not reducible to either a syntactical notion relative to a specified logistic or a semantical notion of the preservation of truth. The author motivates his study of 'proof' by considering Gödel's incompleteness theorem. He gives the usual syntactical and semantical significance to Gödel's famous theorem. Then he goes on to say, "There is a third interpretation ..., namely, that for any formal system S, adequate for the arithmetic of natural numbers, there are correct inferences which cannot be formalized in S," where there is meaningfulness to the idea 'correct inferences' or 'proof' even as a notion of neither syntax nor semantics. The author considers his notion as belonging to "some intermediate discipline which I might call 'apodictics'." This discussion leads him into considerations of absolute provability. Finally, he exhorts the reader to be cautious of conventionalism and to be "prepared to expand our means of proof".

A. A. Mullin (Urbana, Ill.)

2540:

Menger, Karl. An axiomatic theory of functions and fluents. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 454–473. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

In 1944–45, the author made an attempt to analyse the structures of the functions of elementary analysis in algebraic terms, as a system with three operations, addition, multiplication, and substitution. In the first part of this paper, he has returned to this problem. In effect, he proposes to consider the following systems. Let G be an (associative) semigroup with a unit e. Let K be a subsemigroup containing e, and define a partial order on G by $x \leq y$ if and only if there exist elements u, v in K with $yu=x$ and $vy=x$. Assume also that there exist unary operators L and R defined on G such that $L(x)=R(x)=x$ for all $x \in K$ and such that (1) $L(x)x=xR(x)=x$, (2) $L(xy) \leq L(x)$ and (3) $R(xy) \leq R(y)$ for all x and y in G. In addition, one assumes that there is a third unary operator Op such that, for all $x \in G$, $Op(x)x \leq R(x)$ and $xOp(x) \leq L(x)$. An example of such a system can be obtained by choosing G as the set of all mappings f whose range and domain are subsets of a fixed set X; the product is substitution (with careful attention to the domain of the product) and K consists of all restrictions of the identity map of X onto itself. $R(f)$ and $L(f)$ consist of all (x, x) with x chosen from the domain or range of f, respectively. A related paper not mentioned by the author is that by von Neumann [Math. Z. **27** (1928/29), 669–752] in which he bases set theory on the notion of function. The remainder of the paper is devoted to another presentation of the author's views on the notion of function versus fluent. {The reviewer feels that the latter concept is too subjective to form a firm basis for a treatment of the logical structure of science, unless one adopts a completely platonic position, with its attendant statistical problems inherent in a theory of measurement.}

R. C. Buck (Madison, Wis.)

2541:

Nidditch, Peter. A note on the redundant axiom of Principia Mathematica. Mind **69** (1960), 251–252.

The purpose of the paper is to provide a simple demonstration that the fourth axiom of the propositional calculus of Whitehead and Russell's *Principia mathematica* [University Press, Cambridge, England, 1910] is derivable from the other four. The demonstration assumes that the deduction theorem and other elementary properties of '$\vdash$' have been proved. The demonstration provided is not the simplest possible under its assumptions, as it requires the demonstration of $p \lor . q \lor r, p \vdash q \lor . p \lor r$ (12), while a direct demonstration of $p \lor . q \lor r \vdash q \lor . p \lor r$ (25) is no more difficult.

The fifth axiom should read $p \supset q . \supset : r \lor p . \supset . r \lor q$.

P. C. Gilmore (Yorktown Heights, N.Y.)

2542:

Porte, Jean. Schémas pour le calcul des propositions fondé sur la conjonction et la négation. J. Symb. Logic **23** (1958), 421–431.

The author gives five new axiom systems for the propositional calculus in terms of conjunction and negation. Three of these systems have five axioms, one has four, and one has six. To prove their sufficiency he shows that they imply the system with three axioms given by J. B. Rosser in chapter 4 of his book *Logic for mathematicians* [McGraw-Hill, New York, 1953; MR **17**, 935]. The independence of four of the new systems is then proved by the use of a collection of matrices of truth values. The question of the independence of the fifth system is left undecided. The author also compares his axiom systems with some of the standard older systems.

O. Frink (Dublin)

2543:

Ryll-Nardzewski, C. On the categoricity in power $\leqq \aleph_0$. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 545–548. (Russian summary, unbound insert)

The notion of categoricity in a given cardinal number m was introduced by Łoś [Colloq. Math. **3** (1954), 58–62; MR **15**, 845] and Vaught [Nederl. Akad. Wetensch. Proc. Ser. A **57** (1954), 467–472; MR **16**, 208]. This paper is concerned with a necessary and sufficient condition that a theory T, formulated in the first order predicate calculus with equality, be categorical in a cardinal $\leqq \aleph_0$. The condition may be stated as follows. Let α be a formula of T, and let $[\alpha]$ be the class of all formulas equivalent to α. Let F_n be the algebra of all $[\alpha]$ such that α depends only on the variables $x_1, \cdots, x_n$. Then the condition is that F_n be finite. *H. B. Curry* (University Park, Pa.)

2544:

Robinson, Abraham. Obstructions to arithmetical extension and the theorem of Łoś and Suszko. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 489–495.

The author generalizes a result of Łoś and Suszko [Fund. Math. **44** (1957), 52–60; MR **19**, 724] for the pure first order predicate calculus (PC) to arbitrary axiomatic theories formulated in PC. Consider all those increasing sequences of structures M_n such that (the set of axioms) K_0 is satisfied in each M_n and in their union. Call a set K of sentences σ-persistent with respect to K_0 if, for each such sequence, $\bigcup M_n$ is a model for K provided each M_n in the sequence is one. The author shows that each sentence of K is equivalent (on the basis of K_0) to a UE sentence, i.e., in prenex form with a prefix consisting of a string of universal quantifiers followed by a string of existential quantifiers. Łoś and Suszko had proved this result for K_0 empty. *G. Kreisel* (Paris)

2545:

Behmann, Heinrich. Der Prädikatenkalkül mit limitierten Variablen. Grundlegung einer natürlichen exakten Logik. J. Symb. Logic **24** (1959), 112–140.

Let $a \downharpoonleft f$ and $x \upharpoonright (\cdots x \cdots)$ stand for "the predicate f holds for a" and "the predicate of x represented by '$\cdots x \cdots$' ". Then $a \downharpoonleft (x \upharpoonright (\cdots x \cdots))$ is equivalent to the expression obtained by substituting "a" for "x" in "$(\cdots x \cdots)$". The author shows that in certain expressions (for example, Russell's paradoxical predicate

$$[f \upharpoonright (\text{not } (f \downharpoonleft f))] \downharpoonleft [f \upharpoonright (\text{not } (f \downharpoonleft f))])$$

there is a never-ending possibility of making such substitutions. Such expressions are to be considered meaningless. Thus, Russell's paradox is avoided, and this situation is no more surprising to the author than the existence of divergent series in analysis. This approach is formally similar to Church's theory of λ-conversion. (It has the drawback that there is no effective method of finding out whether formulas are meaningful.) Since predicates may not be meaningfully applied to all objects, the author introduces symbolic devices for restricting variables to those objects for which a given predicate has meaning. *E. Mendelson* (New York)

2546:

Myhill, John. Recursive digraphs, splinters and cylinders. Math. Ann. **138** (1959), 211–218.

Let N denote the set of all non-negative integers. A recursively enumerable (r.e.) set S is a splinter if $S = (a, f(a), f^2(a), \cdots)$ for some number a and some recursive function $f(n)$. This notion was introduced by J. Ullian [J. Symb. Logic **23** (1958), 107–108, abstract]. With every recursive function $f(n)$ a digraph (i.e., directed graph) $G(f)$ is associated; it has all non-negative integers as its points and all ordered pairs of the form $(n, f(n))$ as its lines. The sets A and B are isomorphic ($A \cong B$) if there is a recursive permutation of N which maps A onto B. Let $j(x, y) = x + (x+y)(x+y+1)/2$. A set M is a cylinder if $M \cong j(A \times N)$ for some non-empty r.e. set A. This paper studies some relations between splinters, cylinders and the classification of r.e. sets into recursive, creative, simple and mesoic; for the definition of these terms see J. Myhill, Z. Math. Logik Grundlagen Math. **1** (1955), 97–108 [MR **17**, 118]. For the graph-theoretic terminology, see F. Harary, Math. Ann. **138** (1959), 203–210 [MR **22** #18]. Every splinter $\{f^n(a)\}$ included in $G(f)$ lies in exactly one (weak) component, henceforth called the component of the splinter.

Let C be the component of the splinter S; then S is recursive in C and $C - S$ is r.e. Every cylinder is an infinite splinter, but it is not known whether the converse holds. While a splinter cannot be simple (Ullian), every creative set is a splinter and there exist mesoic splinters. The notion of a splinter is generalized to that of an m-n-splinter: a set M is an m-n-splinter if there exist m numbers $a_1, \cdots, a_m$ and n functions $f_1, \cdots, f_n$ such that M is the smallest set which contains $a_1, \cdots, a_m$ and is closed under each of $f_1, \cdots, f_n$. It is proved that every m-1-splinter is an ordinary (i.e., 1-1-) splinter and that every m-n-splinter is an ordinary splinter or a 1-2-splinter; moreover, a set is a 1-2-splinter if and only if it is r.e. The paper closes with a discussion of universal digraphs. It is shown that there exists a digraph U such that every countable digraph is isomorphic (in the digraph-theoretic sense) to a subgraph of U. A stronger result is obtained for the representation of digraphs of the relations $R_1, R_2, \cdots$, where $R_n(x, y) =_{\mathrm{df}} j(x, y) \in W_n$, and $\{W_n\}$ is a canonical enumeration of the class of all r.e. sets.

J. C. E. Dekker (New Brunswick, N.J.)

2547:

Rogers, Hartley, Jr. Recursive functions over well ordered partial orderings. Proc. Amer. Math. Soc. **10** (1959), 847–853.

By a straightforward application of the recursion theorem the author proves the following recursive analogue of a familiar principle for defining continuous functions on the ordinals by transfinite induction. Let W be Spector's set of notations for recursive ordinals [J. Symb. Logic **20** (1955), 151–163; MR **17**, 570] and let $<$ be a well-founded (not necessarily recursive) relation on W. Let $D(x, a)$ be a (not necessarily recursive) relation between numbers x of partial recursive functions Φ_x in some numbering and notations a, and let $f(x, a)$ be a recursive function with the following properties: (i) extensionality, i.e., if $D(x, a)$ and $\Phi_x(\{b|\ b<a\}) = \Phi_y(\{b|\ b<a\})$, then $D(y, a)$; (ii) continuity, i.e., if a is a notation for a minimal element or a limit element of $<$, and if for all $b<a$ we have $D(x, b)$, then $D(x, a)$; and (iii)

if a_1 is the immediate predecessor of a, $D(x, a_1) \to D[f(x, a), a]$ with $\Phi_{f(x,a)} = \Phi_x$ on $\{b \mid b < a\}$. Under (i)–(iii) there is an x_0 such that $D(x_0, a)$ for all a in W. The author observes that this lemma has been used in the literature so often that it is worth formulating it explicitly.

G. Kreisel (Paris)

2548:

Markov, A. A. On the inversion complexity of a system of functions. J. Assoc. Comput. Mach. **5** (1958), 331–334.

Translation of the article in Dokl. Akad. Nauk SSSR **116** (1957), 917–919 [MR **20** #3773].

2549:

Lyndon, R. C. Properties preserved under algebraic constructions. Bull. Amer. Math. Soc. **65** (1959), 287–299.

This is a very useful survey of results (by G. Birkhoff, L. Henkin, A. Horn, H. J. Keisler, J. Łoś, A. I. Mal'cev, A. Robinson, A. Tarski, D. Scott, the author and others) concerning the formal characterisation of elementary sentences which are preserved under algebraic constructions (the passage from an algebraic system to a homomorphic image, the passage from a set of systems to their direct product, etc.).

H. Rasiowa (Warsaw)

SET THEORY

2550:

Szász, Pál. Über den Äquivalenzsatz der Mengenlehre. Mat. Lapok **10** (1959), 49–52. (Hungarian. Russian and German summaries)

This note is an exposition of J. König's proof of Cantor's equivalence theorem.

R. Bott (Cambridge, Mass.)

2551:

Zakon, Elias. On common multiples of transfinite numbers. Canad. Math. Bull. **3** (1960), 31–33.

Ordinals α and β, with $\alpha \geq \beta > 0$, have a common right multiple (i.e., $\tau\alpha = \sigma\beta$ for some τ, σ) if and only if either (i) α is itself a right multiple of β, or (ii) β is a nonlimit ordinal and $\alpha = \beta + c$ for some finite c; when a common right multiple exists, then the least such, μ, has the form $\mu = c\alpha = \alpha + c_1$ with c and c_1 finite. For background, see H. Bachmann, *Transfinite Zahlen* [Springer, Berlin, 1955; MR **17**, 134].

L. Gillman (Rochester, N.Y.)

2552:

Świerczkowski, S. Some remarks on inaccessible alephs. Colloq. Math. **7** (1959), 27–30.

The author calls $\aleph_\mu$ inaccessible if (a) $\sum_{\beta<\alpha} m_\beta < \aleph_\mu$ whenever $\alpha < \omega_\mu$ and $m_\beta < \aleph_\mu$ for $\beta < \alpha$, and (b) $m^n < \aleph_\mu$ for $m, n < \aleph_\mu$. If α is an ordinal number, and s is a sequence of type α of zeros and ones, say $s = (\delta_\beta)_{\beta<\alpha}$ where $\delta_\beta = 0$ or 1, then he defines $\varepsilon|s|\eta = (\delta_\beta)_{\varepsilon\leq\beta<\eta}$ for $\varepsilon < \eta \leq \alpha$; and, if S is a family of sequences of type α of zeros and ones, then $\varepsilon|S|\eta = \{\varepsilon|s|\eta : s \in S\}$, $S|\eta = 0|S|\eta$. If α is a limit ordinal number, a sequence of type α of subsets $(A_\beta)_{\beta<\alpha}$ of a set M is said to be convergent to a subset A of M if to every $x \in M$ there corresponds a $\gamma < \alpha$ such that $x \notin A \dot{-} A_\beta$ for $\gamma < \beta < \alpha$, where $A \dot{-} A_\beta$ denotes the set of all $y \in M$ that belong to precisely one of the sets A, A_β. A function m defined on all subsets of a set M and taking the values 0, 1 is said to be additive with each power less than $\aleph_\mu = \operatorname{card} M$, if $m(\bigcup_{\beta<\alpha} A_\beta) = \sum_{\beta<\alpha} m(A_\beta)$ for mutually exclusive subsets A_β of M and $\alpha < \omega_\mu$.

Under the assumption that card $M = \aleph_\mu$ is inaccessible, the following three statements are known to be unverified hypotheses. (H_1) If to every $\alpha < \omega_\mu$ there corresponds a nonempty family S_α of sequences of type α of zeros and ones, and $S_\alpha|\eta = S_\eta$ wherever $\eta < \alpha$, then there exists a sequence s of type ω_μ of zeros and ones such that $s|\alpha \in S_\alpha$ for every $\alpha < \omega_\mu$. (H_2) Every sequence of type ω_μ of subsets of M contains a convergent subsequence of type ω_μ. (H_3) There exists a zero-one-valued measure m defined on all subsets of M which is additive with each power less than card M and is such that $m(M) = 1$.

Theorem: If card $M = \aleph_\mu$ is inaccessible, then (H_1) and (H_2) are equivalent, and they are implied by (H_3).

F. Bagemihl (Ann Arbor, Mich.)

2553:

Chang, Chen Chung; Morel, Anne C. Some cancellation theorems for ordinal products of relations. Duke Math. J. **27** (1960), 171–181.

Let R be a binary relation, i.e., a set of ordered pairs; the field of R is the set FR of all x such that $(x, y) \in R$ or $(y, x) \in R$. Two binary relations R, S are isomorphic, $R \cong S$, if there exists a one-to-one function f of FR onto FS such that $xRy \Leftrightarrow fxSfy$. The type of R, symbolically τR, is the equivalence class with respect to $\cong$ and containing R; the types are denoted by Greek letters. The equality and ordering of types is done in the usual way. The ordinal product of relations R, S is the relation $R \cdot S$ such that $F(R \cdot S) = FR \times FS$ and $(r_1, s_1) R \cdot S (r_2, s_2) \Leftrightarrow s_1 = s_2$ and $r_1 R r_2$ or $s_1 \neq s_2$ and $s_1 S s_2$. One defines $\tau(R \cdot S) = \tau R \cdot \tau S$.

The aim of the paper is to establish some cancellation properties for ordinal multiplication. Theorem 1: If $\alpha\beta = \gamma\delta$ and if the field of α is finite and $k\alpha = k\gamma$ ($k\alpha$ denoting the cardinality of α), then $\beta = \delta$; in particular, if α is finite and $\neq 0$, then $\alpha\beta = \alpha\gamma \Rightarrow \beta = \gamma$. The right cancellation law is much more complicated; as a synthesis of 15 lemmas, the authors prove the following cancellation law: Let $\alpha, \beta, \gamma, \delta$ be non-zero types satisfying $\alpha\beta = \gamma\delta$; if β, δ are finite, then α, γ are comparable, i.e., $\alpha \leq \gamma$ or $\alpha \geq \gamma$ (theorem 19); in particular, if α, β, γ are non-zero finite types such that $\alpha\beta = \gamma\delta$ and $k\beta = k\delta$, then $\alpha = \gamma$ and $\beta = \delta$. The conjecture C2, stating the comparability of α, γ provided that either β or δ is finite, remains open. The conjecture C4, stating the equality of α, γ for the case that β is finite and equal to δ, remains open as well.

Đ. Kurepa (Zagreb)

2554:

Erdös, P.; Rado, R. Intersection theorems for systems of sets. J. London Math. Soc. **35** (1960), 85–90.

In connection with the Dirichlet's box argument let the following definitions be stated. For cardinal numbers a, b a system Σ_0 of sets X_μ ($\mu \in M$) is called an (a, b)-system provided $|M| = a$, $|X_\mu| = b$ ($\mu \in M$). Σ_0 is called a Δ-system [$\Delta(a)$-system] with a kernel K provided every intersection $X_\mu \cap X_\nu$ ($\mu, \nu \in M$, $\mu \neq \nu$) is the constant

K [and $|M|=a$]. A system Σ_1 of sets Y_ν $(\nu \in N)$ contains another system Σ_0 of sets if for every $\nu \in N$ the member X_ν occurs in Σ_1 at least as many times as X_ν occurs in Σ_0. The box principle states that for every natural number a every $(>a^2, 1)$-system contains a $\Delta(>a)$-system. As generalisations, the following three theorems are established. Theorem I: (1) If $a, b \geqq 1$, then every $(>b^+b^ba^{b+1}, \leqq b)$-system contains a $\Delta(>a)$-system; (2) if $a \geqq 2$, $b \geqq 1$, $a+b \geqq \aleph_0$, then every $(>a^b, \leqq b)$-system contains a $\Delta(>a)$-system. Theorem II: If $a, b \geqq 1$, there exists an (a^{b+1}, b)-system containing no $\Delta(>a)$-system. Theorem III: If $1 \leqq a, b < \aleph_0$ and

$$c = b!a^{b+1}\left(1-\sum_{k=2}^{b}\frac{k-1}{k!a^{k-1}}\right),$$

then every $(>c, \leqq b)$-system contains a $\Delta(>a)$-system. The authors conjecture that it should be possible to replace the factor $b!$ of c by $c_1{}^b$ for some absolute constant c_1; such a result should have applications in the theory of numbers. *Đ. Kurepa* (Zagreb)

2555:

Marcus, Solomon. Sur la représentation d'une fonction arbitraire par des fonctions jouissant de la propriété de Darboux. Trans. Amer. Math. Soc. **95** (1960), 489–494.

The author gives the following lemma. Let X be any set of cardinal a (infinite). Let $\mathscr{F}$ be a family E_λ, $\lambda \in \Lambda$, of subsets of X with cardinal of $\mathscr{F}$ and of each E_λ equal to a. Suppose $2 \leqq n \leqq \aleph_0$. Then X can be partitioned into disjoint subsets $B_{i,\gamma}$, $1 \leqq i < n, \gamma \in \Lambda$, so that every $B_{i,\gamma} \cap E_\lambda$ is not empty (the author quotes an argument given by the reviewer for a special case [Amer. Math. Monthly **57** (1950), 539–540; MR **12**, 399]). From the lemma follows the theorem: if Y is any additive group with cardinal a then every function $f(x)$ from X to Y can be expressed as a sum $f(x)=\sum_{1 \leqq i \leqq n} f_i(x)$, where, for each x, $f_i(x) \neq 0$ for at most 2 values of i, and for each i and each E_λ, $f_i(E_\lambda)=Y$. This generalizes a "topological" result due to Sierpiński of which a special case is the theorem: every $f(x)$ which is real-valued on the real axis is the sum of two functions each having the Darboux property. The lemma is also used to get a decomposition into products $f=\prod f_i$ and a limit representation $f=\lim g_i$, with $f_i(E_\lambda)=(Y-\{0\})$, $g_i(E_\lambda)=Y$ for all i, λ.

I. Halperin (Kingston, Ont.)

COMBINATORIAL ANALYSIS

See also 2546.

2556:

Hall, Marshall, Jr. A survey of combinatorial analysis. Some aspects of analysis and probability, pp. 35–104. Surveys in Applied Mathematics. Vol. 4. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. xi+243 pp. $9.00.

This survey is a marvelous organization of what could have been otherwise regarded as unconnected material. The author's aim has been to illustrate problems and methods (with emphasis on recent research), not to be encyclopedic and not to write a textbook. The survey could nevertheless furnish the basis for an excellent course in combinatorial mathematics until the desired textbook appears.

After a brief historical introduction, the survey is divided into three main sections: problems of enumeration; theorems on choice; existence and construction of designs. The first section considers permutations and combinations, recurrences and generating functions, and partitions. The second section discusses the theorems of Hall and Ramsey and their applications, and relevant techniques from linear programming. The third section treats Latin squares and rectangles, Steiner triple systems, and the construction of (or proof of non-existence of) various partially balanced incomplete block designs, especially the symmetric ones.

In general, proofs are omitted or only hinted, but the general methods are made very clear. Important recent material not available at the time the survey was written includes the disproof of Euler's conjecture on orthogonal latin squares (relevant to section three); the increased understanding of the interplay among flow in networks, the theory of linear inequalities, and theorems of choice (section two); and certain new insights and techniques in the study of block designs (section three).

A. J. Hoffman (New York)

2557:

Bose, R. C.; Shrikhande, S. S. On the construction of sets of mutually orthogonal Latin squares and the falsity of a conjecture of Euler. Trans. Amer. Math. Soc. **95** (1960), 191–209.

Suppose that an integer v possesses u distinct prime factors p_i $(i=1, 2, \cdots, u)$ of multiplicity n_i $(n_i>0)$. If one defines $n(v)=\min(p_1{}^{n_1}, \cdots, p_u{}^{n_u})-1$, then there exist at least $n(v)$ mutually orthogonal Latin squares (MOLS) of order v. If $N(v)$ denotes the maximum number of MOLS of side v, then it is trivial that $N(v) \geqq n(v)$, and it has long been conjectured that $N(v)=n(v)$. This conjecture was disproved by E. T. Parker [Abstract 553–67, Notices Amer. Math. Soc. **5** (1958), 815], and his result cast doubt upon the Eulerian conjecture that $N(v)=n(v)=1$ for the case $v=4t+2$.

In the present paper, which has already become famous, the authors generalize Parker's methods by using certain types of balanced incomplete block designs (BIBD's) to construct sets of MOLS. In particular, they show that if there exists a pairwise balanced design D of index unity and type $(v; k_1, k_2, \cdots, k_m)$, if there exist q_i-1 MOLS of order k_i, and if $q=\min(q_1, q_2, \cdots, q_m)$, then there exist at least $q-2$ MOLS of order v; if D is separable, then there exist at least $q-1$ MOLS. Furthermore, if there exists a BIBD with parameters $v, k, \lambda=1$, then $N(v) \geqq N(k)-1$; if the design is separable, $N(v) \geqq N(k)$.

Various numerical results are obtained; for example, if there exists a BIBD with parameters $v, k, \lambda=1$, then, for $x \leqq k$,

$$N(v-x) \geqq \min[N(k), N(k-1), N(k-x)]-1.$$

Also, the falsity of Euler's conjecture follows at once from the theorem that $N(v) \geqq 2$ for v any positive integer of the form $36m+22$. The paper concludes with an actual construction of 2 MOLS of side 22 and a table giving certain bounds on $N(v)$ for $v \leqq 150$.

R. G. Stanton (Waterloo, Ont.)

2558:

Hanani, Haim. On quadruple systems. Canad. J. Math. **12** (1960), 145–157.

Given a set E of n elements, denote by $S(l, m, n)$,

$l \leq m \leq n$, a system of subsets of E having m elements each, such that every subset of E having l elements is contained in exactly one set of the system $S(l, m, n)$. A Steiner triple system is an $S(2, 3, n)$. It has been known since 1859 [M. Reiss, J. Reine Angew. Math. **56** (1859), 326–344] that the obviously necessary condition $n \equiv 1$ or $3 \pmod 6$ is also sufficient. This paper proves the comparable results for quadruple systems $S(3, 4, n)$, namely, that the obviously necessary condition $n \equiv 2$ or $4 \pmod 6$ is also sufficient. The proof is comparable to the earlier proof by Reiss in that the method is constructive, based on auxiliary sets of pairs and devices for constructing larger systems from smaller systems whose existence is given by induction or in the special cases $n = 14$ or 38 by direct construction. *Marshall Hall, Jr.* (Pasadena, Calif.)

2559:

van der Corput, J. G. On an identity of Block and Marschak. Bull. Amer. Math. Soc. **66** (1960), 28–31.

The following theorem is proved: If p, q and n denote integers with $0 \leq p \leq q \leq n$ and $n \geq 1$, and if $u_1, u_2, \cdots, u_n$ are any numbers, then

$$\sum{}^1 \prod_{k=1}^{n} \Big(\sum_{j=k}^{n} u_{s_j} \Big)^{-1} = \Big[\Big(\prod_{i=p+1}^{n} u_i \Big) \Big(\prod_{k=1}^{p} \sum_{j=k}^{q} u_j \Big) \Big]^{-1},$$

where $\sum^1$ is extended over the permutations $(s_1, s_2, \cdots, s_n)$ of $(1, 2, \cdots, n)$ which rank 1 before 2, 2 before 3, $\cdots$, $p-1$ before p, and finally p before each of the numbers $p+1, p+2, \cdots, q$. This generalizes the identity cited in the title, which is the special case $p = 1$. *H. D. Block* (Ithaca, N.Y.)

2560:

Harary, Frank. Some historical and intuitive aspects of graph theory. SIAM Rev. **2** (1960), 123–131.

The author begins with Euler's discovery that, for a connected graph, a circuit simply covering all the edges is possible if and only if the valency at each vertex is even. He quotes many other theorems, such as Fary's (every planar graph can be drawn in a plane so that each of its edges is straight), Frucht's (for any finite group there is a trivalent graph having the given group for its group of automorphisms), and Whitney's (every graph is characterized by its "line graph", except when the latter is a triangle). After exhibiting a planar, bridgeless, trivalent graph that is not Hamiltonian, he remarks that Hamilton, in the process of analysing his "round the world" game, became one of the founders of the theory of abstract groups. The reader is left with the problem of deciding whether every map with 39 states can be colored with 4 colors. *H. S. M. Coxeter* (Toronto, Ont.)

2561:

Harary, Frank; Norman, Robert Z. Dissimilarity characteristic theorems for graphs. Proc. Amer. Math. Soc. **11** (1960), 332–334.

A block of a graph is defined as a maximal connected subgraph containing no cut-points of itself. Two points [lines, blocks] are called similar with respect to the subgroup $\mathfrak{A}$ of the automorphism group of the graph if a permutation of $\mathfrak{A}$ sends one of them into the other. The author states that if G is a connected graph with n similarity classes of blocks with respect to $\mathfrak{A}$, p the number of equivalence classes of points of G similar with respect to $\mathfrak{A}$, and p_k the number of equivalence classes of similar points of the blocks of the kth similarity class, with respect to $\mathfrak{A}$, then $p - 1 = \sum_{k=1}^{n} (p_k - 1)$. For a proof the reader is referred to G. W. Ford, R. Z. Norman and G. E. Uhlenbeck [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 203–208; MR **17**, 1231]. A result of Otter on trees [R. Otter, Ann. of Math. (2) **49** (1948), 583–599; MR **10**, 53], an extension of this result to Husimi trees [F. Harary and R. Z. Norman, ibid. **58** (1953), 134–141; MR **14**, 1113], and a theorem of the first author [Amer. Math. Monthly **66** (1959), 405–407; MR **21** #2986] are deduced. *G. A. Dirac* (Hamburg)

2562:

Nettleton, R. E.; Green, M. S. Moebius function on the lattice of dense subgraphs. J. Res. Nat. Bur. Standards Sect. B **64B** (1960), 41–47.

A subgraph G' of a connected graph G consists of a subset of the points of G together with all lines joining any two points in the subset. Let $G(G')$ be the subgraph of G containing those points in $G - G'$ which have neighbors in G'. A connected subgraph G' is called k-dense if $G - G'$ has at most $k - 1$ points not in $G(G')$. It was shown in a previous paper by Nettleton [unpublished] that the k-dense subgraphs of G together with the null graph ϕ form a lattice. A minimal subgraph G' whose removal disconnects G is called an "isthmus" if G' is itself a complete graph, and is an "articulator" otherwise.

The Moebius function f_k on this lattice is defined and is calculated for graphs G containing isthmuses and articulators. f_1 evaluated for the null graph ϕ is shown to vanish if G contains an isthmus, while for any integer q there exist graphs containing articulators for which $f_1(\phi) = q$. *F. Harary* (Ann Arbor, Mich.)

2563:

Izbicki, Herbert. Über die explizite Automorphismengruppe von speziellen Graphen. J. Reine Angew. Math. **203** (1960), 35–39.

An automorphism of a graph G is a permutation of its vertices which preserves the number of links joining any two and the number of loops on any one. A "special" graph K is one without loops and with no multiple join between two vertices. Let $\{K_1, K_2, \cdots, K_r\}$ be a partition of the vertices of a special graph K into as few classes K_i as possible so that the following conditions are satisfied: any two members of the same class K_i are joined by an edge, and if $K_i \neq K_j$ either every vertex of K_i is joined to every vertex of K_j or there are no joins between K_i and K_j. The author defines a graph K^* with vertices $k_1, k_2, \cdots, k_r$ so that there are as many loops on k_i as there are vertices in K_i and the number of edges joining k_i and k_j (where $i \neq j$) is 1 or 0 according as there are or are not edges joining K_i and K_j in K.

The author expresses the automorphism group of K in terms of that of K^* and the groups of permutations on the sets K_i. *W. T. Tutte* (Toronto)

2564:

Rapaport, Elvira Strasser. Cayley color groups and Hamilton lines. Scripta Math. **24** (1959), 51–58.

Let G be a finite group with generators $g_1, g_2, \cdots, g_s$. Then there exists a regular directed graph of degree s which

is a subgraph of the Cayley colour group [Hilton, *An introduction to the theory of groups of finite order*, Clarendon, Oxford, 1908] and corresponds to the generators $g_1, \cdots, g_s$. If all these generators are involutory, the graph can be considered as non-directed. The author investigates the existence of Hamilton lines of the graphs of such a group. The main result: Let S_n $(n \geqq 3)$ be the symmetric group, $t(i)$ $(i=1, \cdots, n-1)$ the transpositions $(i, i+1)$. Let $g_1=t(1)$, $g_2=t(1)t(3)\cdots t(2[n/2]-1)$, $g_3=t(2)t(4)\cdots t(2[(n-1)/2])$. Then the elements g_1, g_2, g_3 generate S_n and the corresponding graph has a Hamilton line.

M. Fiedler (Prague)

ORDER, LATTICES

See also 2548, 2569.

2565:

Michael, E. A class of partially ordered sets. Amer. Math. Monthly **67** (1960), 448–449.

The results have appeared several times, according to the editor's Note in same Monthly **67** (1960), 779; first appearance in Higman, Proc. London Math. Soc. (3) **2** (1952), 326–336 [MR **14**, 238]; see also #2566 below.

J. Isbell (Seattle, Wash.)

2566:

Kruskal, J. B. Well-quasi-ordering, the Tree Theorem, and Vazsonyi's conjecture. Trans. Amer. Math. Soc. **95** (1960), 210–225.

If Q is a set, a tree over Q is defined to be a finite tree, in the sense of combinatorial topology, in which (i) one vertex is chosen as root, (ii) at each vertex, the edges incident with it, and pointing away from the root, are assigned a linear order, and (iii) to each vertex v is assigned an element $f(v)$ of Q. A quasiorder in Q induces a quasi-order in $T(Q)$, the set of trees over Q, in which $t_1 \leqq t_2$ if there is a homeomorphism of t_1 into t_2 which maps the root onto the root, which preserves the order among edges whenever it is defined, and such that if the vertex v is mapped onto v', then $f(v) \leqq f(v')$. The main theorem of the paper is that if the order in Q is a partial well order, so is that in $T(Q)$. It includes, or has as easy consequences, all known theorems on partial well order, except those of Erdös and Rado [J. London Math. Soc. **34** (1959), 222–224; MR **21** #2604] on infinite vectors.

Graham Higman (Chicago, Ill.)

2567:

Matsushima, Yatarô. Hausdorff interval topology on a partially ordered set. Proc. Amer. Math. Soc. **11** (1960), 233–235.

In a partially ordered set X, suppose that for each x the set of elements incomparable with x has a finite subset S such that every element of X is comparable with x or with some element of S. Then the interval topology of X is Hausdorff.

J. Isbell (Seattle, Wash.)

2568:

Wolk, E. S. On partially ordered sets possessing a unique order-compatible topology. Proc. Amer. Math. Soc. **11** (1960), 487–492.

A partially ordered set X is said to have a unique order-compatible topology if (1) the order and interval topologies coincide. Evidently this implies (2) every diverse set (set of pairwise incomparable elements) is closed in the interval topology. The author raises the question whether (2) implies (1), and shows that it does in case X is first-countable in the interval topology. He goes on to investigate order-theoretic conditions for first-countability and for (2), obtaining a sufficient condition for second-countability and asserting a necessary and sufficient condition for (2). The latter is wrong, refuted by an infinite totally unordered set.

J. Isbell (Seattle, Wash.)

GENERAL MATHEMATICAL SYSTEMS

See also 2549, 2658a.

2569:

Marczewski, E. Independence in algebras of sets and Boolean algebras. Fund. Math. **48** (1959/60), 135–145.

Let (A, F) be an abstract algebra, A being the set of elements over which each fundamental operation f in F acts. Let $A^{(n)}$ be the class of algebraic operations, that is, the smallest class of functions containing the n identity functions and closed under composition with the fundamental operations. A subset N of A is called a set of independent elements whenever, for each sequence $a_1, \cdots, a_n$ of different elements of N and each two functions f and g in $A^{(n)}$, if $f(a_1, \cdots, a_n)=g(a_1, \cdots, a_n)$ then f and g are identical in A. The author considers independence of elements in the following five algebras (of subsets of a set B): $(B; \cup, ')$, $(B; \cup, \setminus)$, $(B; \cup, \cap)$, $(B; \cup)$, and $(B; \cap)$. The symbol $\setminus$ is set subtraction. For each of the algebras, the author gives necessary and sufficient conditions for the sets $E_1, \cdots, E_n$ to be independent. He also discusses the relation of independence between the various algebras.

S. Ginsburg (Santa Monica, Calif.)

2570:

Urbanik, K. A representation theorem for Marczewski's algebras. Fund. Math. **48** (1959/60), 147–167.

Using the notation in the above review, for $1 \leqq k \leqq n$ let $A^{(n,k)}$ be the subclass of $A^{(n)}$ which contains all functions depending on at most k variables. Let $A^{(n,0)}$ be the subclass of $A^{(n)}$ containing all constant functions. An algebra is called a Marczewski algebra if for every pair of integers j, n $(1 \leqq j \leqq n)$ and every two functions f, g in $A^{(n)}$ for which the equality $(*)\, f(x_1, \cdots, x_n)=g(x_1, \cdots, x_n)$ depends on x_j there exists a function h in $A^{(n-1)}$ so that equality $(*)$ is equivalent to $x_j=h(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$. The central result of the paper is the following theorem. Let (A, F) be a Marczewski algebra, F consisting of all algebraic operations in the algebra. (i) If $A^{(0)} \neq 0$ and $A^{(3)} \neq A^{(3,1)}$, then there is a field K such that A is a linear space over K. Further, there exists a subspace A_0 of A such that F is the class of all functions f defined as $f(x_1, \cdots, x_n)=\sum_{k=1}^{n} \lambda_k x_k + a$, where λ_i is in K and a is in A_0. (ii) If $A^{(0)}=0$ and $A^{(3)} \neq A^{(3,1)}$, then there is a field K such that A is a linear space over K. Further, there is a subspace A_0 of A such that F is the class of all functions f defined as $f(x_1, \cdots, x_n)=\sum_{k=1}^{n} \lambda_k x_k + a$, where λ_i is in K, $\sum_{k=1}^{n} \lambda_k=1$, and a is in A_0. (iii) If $A^{(3)}=A^{(3,1)}$, then

there is a group G of transformations of the set A such that every transformation that is not the identity has at most one fixed point in A. Moreover, there is a subset A_0 of A normal with respect to the group G such that F is the class of all functions f defined as $f(x_1, \cdots, x_n) = g(x_j)$ $(1 \leq j \leq n)$ or $f(x_1, \cdots, x_n) = a$, where g is in G and a is in A_0. (A subset A_0 of A is said to be normal with respect to the group G if A_0 contains fixed points of all transformations that are not the identity belonging to G and $g(A_0)$ is a subset of A_0 for every g in G.)

S. Ginsburg (Santa Monica, Calif.)

2571:

Ádám, András. On the definitions of direct product in universal algebra. Publ. Math. Debrecen **6** (1959), 303–310.

The direct decompositions of an abstract algebra A can be characterized internally by sets $\{\rho_\lambda\}$ of congruences on A with the property that the relations $x \equiv a_\lambda \pmod{\rho_\lambda}$ always have a unique solution. A is then the unrestricted direct product of its quotient algebras A/ρ_λ. In case A has a one-element subalgebra contained in all subalgebras, there are two other internal characterizations: (i) in terms of subalgebras A_λ of A; and (ii) in terms of orthogonal idempotent endomorphisms ε_λ of A. The author establishes the appropriate one-one correspondences between these three characterizations. An example, due to L. Fuchs and G. Szász, is given disproving the theorem of G. Birkhoff [*Lattice theory*, Amer. Math. Soc., New York, 1948; MR **10**, 673; p. 87, Theorem 4] that the finite direct decompositions of A are in one-one correspondence with the sets of permutable congruences $\theta_1, \theta_2, \cdots, \theta_r$ such that $\bigcap_{i=1}^r \theta_i = O$ and $(\theta_1 \cap \theta_2 \cap \cdots \cap \theta_{i-1}) \cup \theta_i = I$ $(i = 2, 3, \cdots, r)$. This theorem is correct in the context of Chapter VI, §§ 2–3 (algebras all of whose congruences are permutable), but its statement suggests that a wider context is intended.

P. J. Higgins (London)

THEORY OF NUMBERS

See also B3114, B3115.

2572:

Sierpiński, Wacław. ★Teoria liczb. Część 2. [Theory of numbers. Part 2.] Monografie Matematyczne, Tom 38. Państwowe Wydawnictwo Naukowe, Warsaw, 1959. 487 pp. zł 80.00.

"Nearly ten years have elapsed since the third edition of *Teoria liczb*, which appeared as the nineteenth volume of Monografie Matematyczne [Warsaw, Wrocław, 1950; MR **13**, 821]. In that time, many new interesting results in elementary number theory have been found and new problems proposed... As an example of progress, the largest prime number known in 1950 was $2^{127}-1$; to-day, the largest prime number is $2^{3217}-1$. 12 perfect numbers were known then, to-day 18. During the last ten years a dozen or so new composite Fermat numbers were found, among them $2^{2^{10}}+1$, $2^{2^{16}}+1$, $2^{2^{1945}}+1$... This book intends to acquaint the reader with the interesting results and problems of elementary number theory which were not discussed in the 1950 edition. It is therefore a completion rather than a continuation of the 1950 book ..." (from the Introduction).

Chapter 1 deals with divisibility and linear forms. Chapter 2 concerns equations of second degree and higher, discusses among others the equations $x^2+x+1=3y^2$, $x^3+y^3=2z^3$, triangular numbers, the equation $x^3+y^3=z^3+w^3$ in rational numbers, etc. Chapter 3, theorems of Fermat, Euler and Wilson, including discussion of perfect and amicable numbers. Chapter 4, the function $E(x)$, including Ramanujan's theorem. Chapter 5, congruences. Chapter 6, continued fractions and development on a given basis. Chapter 7, prime numbers, including a discussion of Goldbach's hypothesis, numbers of the form $n!$, prime numbers in quadratic forms, Mersenne's and Fermat's numbers, theorems of H. J. Scherk, E. M. Wright and H. E. Richart. Chapter 8, problems from additive number theory, including Schur's theorem, magic squares. Chapter 9, decomposition of integers into sums of powers of integers.

This excellent volume contains the first comprehensive account of the new results obtained on electronic computing machines and many new recent results and problems in elementary number theory. The book can be regarded as complementing Dickson's monumental *History of the theory of numbers* [Stechert, New York, 1934]. The elegant and erudite presentation contains proofs of most of the very numerous results mentioned.

S. M. Ulam (Los Alamos, N.M.)

2573:

Moroń, Z. On almost-perfect decompositions of rectangles. Wiadom. Mat. (2) **1** (1955/56), 175–179. (Polish)

The author continues his investigations [Wiadom. Mat. (2) **1** (1955), 75–94; MR **16**, 1046] into decompositions of rectangles into squares. He calls such a decomposition almost perfect of the first, second or third kind according as any two congruent squares in the decomposition have no point in common, or have at most a vertex in common, or have at most part of an edge in common. He then defines in a natural way the notion of a basis for almost perfect decompositions, and obtains results about the lengths of minimal bases corresponding to the three different kinds of almost perfect decomposition.

H. Halberstam (London)

2574:

McCarthy, P. J. The generation of arithmetical identities. J. Reine Angew. Math. **203** (1960), 55–63.

Let $(a, b)_k$ denote the largest integral kth power which divides both a and b. (When $k=1$, this is merely the greatest common divisor of a and b.) A complex-valued function $f(n, r)$ of two integral variables is called a k-even function of n mod r if $f(n, r) = f((n, r^k)_k, r)$, and the class of all such functions, for a fixed r, is denoted by E_k. In this paper the author characterizes the class E_k, thus generalizing some of the work of Eckford Cohen [Duke Math. J. **25** (1958), 401–421; MR **20** #5756] who treated the case $k=1$ in great detail. The author uses these functions to obtain new proofs of some classical arithmetical identities and to derive a number of new ones.

T. M. Apostol (Pasadena, Calif.)

2575:

Cohen, Eckford. Arithmetical functions of a greatest common divisor. I. Proc. Amer. Math. Soc. **11** (1960), 164–171.

Let (m, n) denote the greatest common divisor of m and n; let α be a real number; let $g(n)$ be a bounded arithmetic function; and write $f_\alpha(n) = \sum_{d\delta=n} g(d)\delta^\alpha$. Using methods familiar in the elementary theory of numbers, the author establishes the asymptotic formula (for $x \to \infty$)

$$\sum_{a,b \le x} f_\alpha((a, b)) = \frac{L(\alpha+1, g)}{(\alpha+1)\zeta(\alpha+1)}\{2\zeta(\alpha)-\zeta(\alpha+1)\}x^{\alpha+1}+O\{E_\alpha(x)\},$$

where $L(\alpha+1, g) = \sum_{n=1}^\infty g(n)/n^{\alpha+1}$ and $E_\alpha(x)$ is x^α, $x^2 \log x$, or x^2 according as $\alpha > 2$, $\alpha = 2$, or $1 < \alpha < 2$. He also obtains an asymptotic formula, with error term $O(x^{3/2} \log x)$, for the case $\alpha = 1$. The paper concludes with the discussion of a number of special cases. {Reviewer's remark. The factor x in the equation (3.4) should be replaced by x^2.}

L. Mirsky (Sheffield)

2576:

Babaev, G. Distribution of integral points on certain surfaces defined by norms. Dokl. Akad. Nauk SSSR **134** (1960), 13–15 (Russian); translated as Soviet Math. Dokl. 1, 992–995.

Theorems are proved concerning the asymptotic behavior of the distribution of integral points on certain ellipses, hyperbolas, and surfaces of the third order as these curves or surfaces tend to infinity.

It is proved, for example, that if $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$, if $\delta = 1$ for $-d \equiv 2$, 3 mod 4 and $\delta = 4$ for $-d \equiv 1$ mod 4, if the Legendre symbol $\left(\frac{-d}{p_i}\right) = 1$ for the prime factors p_i $(i = 1, \cdots, s)$ of the odd integer m, if r is the number of integral points on the ellipse $x^2 + dy^2 = \delta m$, and if T is the number of integral points on this ellipse contained between rays from the origin inclined at angles a and b to the x-axis, then if $\Delta = \sqrt{(\ln (p_1 \cdots p_s)/\ln r)} \to 0$ for $m \to \infty$, it follows that

$$T = (r/2\pi)\{\arctan(d^{1/2}\tan b) - \arctan(d^{1/2}\tan a)\} + O(r).$$

F. Goodspeed (Quebec)

2577:

Hodges, John H. Some determinantal equations over a finite field. Math. Z. **72** (1959/60), 355–361.

Let A be a nonsingular matrix of order m with elements in GF(q) and let $1 \le t \le m$. The author determines the number of solutions of the determinantal equation

$$(*) \qquad \begin{vmatrix} A & V \\ U & 0 \end{vmatrix} = \beta,$$

where β is an assigned number of GF(q) and U, V are $t \times m$ and $m \times t$ matrices, respectively. For example, for $\beta \ne 0$, the number of solutions of (*) is equal to $q^{t(m-t)}g_t g(m, t, t)/(q-1)$, where $g(m, t, t) = \prod_{i=0}^{t-1} (q^m - q^i)$, $g_t = g(t, t, t)$. Moreover for $\beta = 0$ the number of solutions with U, V of rank t is found. Next the generalized equation

$$\sum_{i=0}^{k} \begin{vmatrix} A_i & U_i \\ U_i & 0 \end{vmatrix} = \beta,$$

where k is a fixed integer ≥ 1, is considered. As an application of the results for this equation, the author evaluates certain generalized Gauss sums.

It is remarked that some of the methods of the paper are similar to those employed by the reviewer [Math. Nachr. 11 (1954), 135–142; Arch. Math. 5 (1954), 19–31; MR **15**, 778, 777] in discussing analogous problems for symmetric and skew matrices.

L. Carlitz (Durham, N.C.)

2578:

Carlitz, L. Some cyclotomic matrices. Acta Arith. **5**, 293–308 (1959).

The five principal classes of matrices considered in this paper are the following. (i) $(e^{2\pi i r s/n})$ $(r, s = 0, 1, \cdots, n-1)$; (ii) $(a + b\chi(r) + c\bar\chi(s) + d\chi(r)\bar\chi(s))$, where a, b, c, d are constants, $\chi(r)$ is an arbitrary character (mod n), and r, s run through the numbers of a reduced residue system (mod n) in some prescribed order; (iii) $(c + \chi(\alpha + r + s))$ $(r, s = 1, \cdots, p-1)$, where c is arbitrary but α is an integer; (iv) $(c + \chi(r-s))$ $(r, s = 1, \cdots, p-1)$; (v) $(\chi(s-r))$ $(r, s = 0, 1, \cdots, n-1)$.

The order of (ii) is $\phi(n)$, whereas the order of (iii) is $p-1$, where p is a prime. In each case the characteristic roots are determined, although for (iii) the results are not entirely explicit. It is shown, for example, that if $\chi(r)$ is a non-principal character (mod n), then the characteristic roots of the matrix (v) are the numbers $e^{2\pi i r/n}\tau(\chi)$ $(r = 0, 1, \cdots, n-1)$, where $\tau(\chi) = \sum_{s=1}^{n-1} \chi(s)e^{2\pi i s/n}$. The method for dealing with (i) is due to Schur [see E. Landau, *Vorlesungen über Zahlentheorie*, Vol. 1, S. Hirzel, Leipzig, 1927; p. 162]. Matrices (ii) and (iii) generalize two classes of matrices previously considered by D. H. Lehmer [Pacific J. Math. 6 (1956), 491–499; MR **19**, 7].

A. L. Whiteman (Princeton, N.J.)

2579:

Carlitz, L. Some finite summation formulas of arithmetic character. II. Acta Math. Acad. Sci. Hungar. **11** (1960), 15–22. (Russian summary, unbound insert)

Extending results of his earlier paper [Publ. Math. Debrecen 6 (1959), 262–268; MR **22** #1549] the author derives a number of arithmetical summation theorems of which the following is representative. Let $n \ge 1$; $e_1, \cdots, e_n \ge 1$; let ζ_i denote a primitive e_ith root of unity. Let $f_i(x, \zeta_i)$ be functions that satisfy:

$$(1) \qquad f_i(x+1, \zeta_i) = \zeta_i^{-1}f_i(x, \zeta_i) \quad (i = 1, \cdots, n),$$

$$(2) \qquad \sum_{r=0}^{k-1} \zeta_i^r f_i(x + r/k, \zeta_i) = C_i^{(k)} f_i(kx, \zeta_i),$$

where $C_i^{(k)}$ is independent of x and $k \equiv 1$ (mod e), where e is the least common multiple of $e_1, \cdots, e_n$. Also let $a_1, \cdots, a_n$ be positive integers that are relatively prime in pairs and such that $a_i \equiv 1$ (mod e) $(i = 1, \cdots, n)$; put $A = a_1 a_2 \cdots a_n$. Then we have

$$\sum_{r=0}^{kA-1} \zeta_1^r\zeta_2^r \cdots \zeta_n^r f_1(x_1 + r/a_1k, \zeta_1) \cdots f_n(x_n + r/a_nk, \zeta_n) =$$

$$C \sum_{r=0}^{k-1} \zeta_1^r\zeta_2^r \cdots \zeta_n^r f_1(a_1x_1 + r/k, \zeta_1) \cdots f_n(a_nx_n + r/k, \zeta_n),$$

where $C = C_1^{(a_1)}C_2^{(a_2)} \cdots C_n^{(a_n)}$. Relations (1) and (2) are suggested by certain functional equations for the Euler polynomials and functions.

A. L. Whiteman (Princeton, N.J.)

2580:

Al-Salam, Waleed A. q-Bernoulli numbers and polynomials. Math. Nachr. **17**, 239–260 (1959).

Two well-known q-analogs of the exponential function e^x are

$$e(x) = 1+\sum_{n=1}^{\infty}\frac{(1-q)^n x^n}{(1-q)(1-q^2)\cdots(1-q^n)},$$

$$E(x) = 1+\sum_{n=1}^{\infty}\frac{q^{n(n-1)/2}(1-q)^n x^n}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Both $e(x)$ and $E(x)\to e^x$ as $q\to1$. In this paper there are discussed q-analogs of the Bernoulli numbers and polynomials as well as q-analogs of various relations involving the ordinary Bernoulli polynomials. For example, one kind of q-Bernoulli polynomial is defined by means of

$$\frac{te(tx)}{e(t)-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{[n]!},$$

where $[n]=(q^n-1)/(q-1)$, $[n]!=\prod_{j=1}^{n}[j]$ $(n\geq1)$, $[0]!=1$. A second kind is defined by means of

$$\frac{tE(tx)}{E(t)-1} = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{[n]!}q^{n(n-1)/2}.$$

Ideas of Nörlund [*Vorlesungen über Differenzenrechnung*, J. Springer, Berlin, 1924] are employed to generalize the q-Bernoulli polynomials to polynomials of higher order. Included among the applications are a q-analog of the Euler-Maclaurin theorem and several properties of "q-Appel sets" of polynomials.

A. L. Whiteman (Princeton, N.J.)

2581:

Shapiro, Harold N. Tauberian theorems and elementary prime number theory. Comm. Pure Appl. Math. **12** (1959), 579–610.

A. Selberg proved the celebrated formula

$$(1)\qquad \sum_{p\leq x}(\log p)^2+\sum_{pq\leq x}\log p\log q = 2x\log x+O(x)$$

in an elementary way and Selberg [Ann. of Math. (2) **50** (1949), 305–313; MR **10**, 595] and the reviewer [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 374–385; MR **10**, 595] deduced the prime number theorem $\vartheta(x)=x+o(x)$ from (1) in an elementary way. To analyse the relation of (1) and the prime number theorem the reviewer proved [J. Indian Math. Soc. (N.S.) **13** (1949), 131–144; MR **11**, 420] that if $1<p_1<p_2<\cdots$ is any sequence of real numbers satisfying (1) then

$$(2)\qquad \frac{1}{x}\sum_{p_i\leq x}\log p_i = 1+o(1)$$

(i.e., it was not assumed that $\sum_{p_i\leq x}\log p_i>cx$ as in the proof of the prime number theorem). The proof of (2) was very complicated. The author first of all gives a simpler proof of (2).

The reviewer also asked the following question: Assume that $1<p_1<p_2<\cdots$ is any sequence of real numbers which satisfies (1) with an error term $o(x\log x)$. Does this weakened form of (1) also imply (2)? The reviewer was unable to answer this question. The author answers this question affirmatively; his proof is elementary but fairly complicated.

Some other related questions are also discussed.

P. Erdős (Budapest)

2582:

Chih, Tsung-tao. A divisor problem. Acad. Sinica Science Record **3** (1950), 177–182.

Let $d(n)$ be the number of divisors of n and put $D(x)=\sum_{n\leq x}d(n)$. In this paper the following two theorems are proved. Theorem 1: Let $\phi(x)$ be positive and increasing and $\phi(x)/x$ decreasing for $x>0$. Further suppose that $\phi(x)/x\to0$ and $\phi(x)/\log^2 x\to\infty$ as $x\to\infty$. Then we have for almost all $x>0$: $D(x+\phi(x))-D(x)\sim\phi(x)\log x$. Theorem 2: Let $\phi(x)$ be positive and increasing and $\phi(x)/x$ decreasing for $x>0$. Further suppose that $\phi(x)/x\to0$ and $\phi(x)/\log^6 x\to\infty$ as $x\to\infty$. Then we have for almost all $x>0$:

$$D(x+\phi(x))-D(x) = \phi(x)\log x+2\gamma\phi(x)+o(\phi(x)),$$

where γ is Euler's constant. The method employed is essentially that used by A. Selberg [Arch. Math. Naturvid. **47** (1943), no. 6, 87–105; MR **7**, 48] to study the normal density of primes in small intervals, and the difference between consecutive primes.

A. L. Whiteman (Princeton, N.J.)

2583:

Chih, Tsungtao. The Dirichlet's divisor problem. Sci. Rep. Nat. Tsing Hua Univ. Ser. A **5** (1950), 402–427.

Let $d(n)$ denote the number of divisors of n. Dirichlet's divisor problem is concerned with the estimation of

$$\Delta(x) = \sum_{n\leq x}d(n)-x\log x-(2\gamma-1)x,$$

where γ is Euler's constant. Employing Titchmarsh's [Proc. London Math. Soc. (2) **38** (1934), 96–115] two-dimensional version of Van der Corput's method, the author proves that $\Delta(x)=O(x^{15/46}\log^{10/9}x)$. Because of the appearance of hyperbolic rather than circular sectors the analysis is essentially more intricate than in the circle problem. A difficult point, which involves the possible vanishing of a certain Hessian, is overcome by following a method due to Min [Trans. Amer. Math. Soc. **65** (1949), 448–472; MR **11**, 84].

{Note: To the best of the reviewer's knowledge the result in this 1950 paper has not subsequently been superseded. In 1953 H. E. Richert [Math. Z. **58** (1953), 204–218; MR **15**, 11] proved independently that

$$\Delta(x) = O(x^{15/46}\log^{30/23}x).$$

Other important related papers published after 1950 are: R. A. Rankin [Quart. J. Math. Oxford Ser. (2) **6** (1955), 147–153; MR **17**, 240]; Yüh, Ming-i [Sci. Record (N.S.) **2** (1958), 326–328; MR **21** #35]; Yin, Wen-lin [ibid. **3** (1959), 6–8; MR **21** #1287].}

A. L. Whiteman (Princeton, N.J.)

2584:

Knapowski, S. On the mean values of certain functions in prime number theory. Acta Math. Acad. Sci. Hungar. **10** (1959), 375–390. (Russian summary, unbound insert)

Let $\Delta(x)=\sum_{n\leq x}\Lambda(n)-x$, where $\Lambda(n)=\log p$ if $n=p^m$ $(m=1,2,\cdots)$ and zero otherwise. Let $M(x)=\sum_{n\leq x}\mu(n)$, where μ is the Möbius function. The author gives lower estimates for the integrals $\int_1^T x^{-1}|\Delta(x)|dx$ and $\int_1^T x^{-1}|M(x)|dx$. The results are: (1) If $\rho_0=\beta_0+i\gamma_0$

($\beta_0 \geqq \frac{1}{2}$) is a zero of Riemann's Zeta-function, then

$$\int_1^T x^{-1}|\Delta(x)|dx > T^{\beta_0} \exp\left\{-\frac{14 \log T}{(\log \log T)^{\frac{1}{2}}}\right\}$$

for $T \geqq \max\{c_1, \exp(\exp 60 \log^2 |\rho_0|)\}$, where c_1 is a numerical constant. (2) If $\int_1^T x^{-1}|M(x)|dx < aT^{1/2}$, for $T \geqq 1$, a being independent of T, then

$$\int_1^T x^{-1}|M(x)|dx > T^{\frac{1}{2}} \exp\left\{-\frac{\log T}{(\log \log T)^{\frac{1}{2}}}\right\}$$

for $T > \max(c_2, e^a)$, where c_2 is a numerical constant.

K. Chandrasekharan (Bombay)

2585:

Knapowski, S. On an explicit lower estimate in prime number theory. J. London Math. Soc. **34** (1959), 437–441.

Let $\psi(X, k, l)$ be the sum $\sum_n \Lambda(n)$ ($n \equiv l \pmod{k}$, $n \leqq X$), where k and l are relatively prime positive integers. The following theorem is proved: there exists a positive constant c_0 such that, if $T \geqq \max(c_0, \exp(\exp(\log k)^3))$, then

$$\max_{1 \leqq X \leqq T} |\psi(X, k, l) - \psi(X, k, 1)| > T^{\frac{1}{2}} \exp\left(-\frac{\log T}{(\log \log T)^{\frac{1}{2}}}\right).$$

The result and method of proof are similar to the author's previous paper, Acta Arith. **4** (1958), 57–70 [MR **20** #3105].

N. C. Ankeny (Cambridge, Mass.)

2586:

Shanks, Daniel. A note on Gaussian twin primes. Math. Comput. **14** (1960), 201–203.

The Gaussian integers $n-1+i$ and $n+1+i$ are a pair of Gaussian twin primes whenever $(n-1)^2+1$ and $(n+1)^2+1$ are rational primes. Let $g(N)$ be the number of such pairs for $4 \leqq n+1 \leqq N$. The author determines $g(N)$ for $N = 500(500)18500$ from a table due to him giving the largest prime factor of n^2+1 for $n = 1(1)18500$. The author notes that the numerical data so obtained bear out the truth of the conjectured relationship

$$g(N) \sim c \int_2^N \frac{dx}{(\log x)^2},$$

$$c = \frac{\pi^2}{8} \prod_{p \equiv 1 \pmod 4} \left(1-\frac{4}{p}\right)\left(\frac{p+1}{p-1}\right)^2 = 0.48762\cdots.$$

M. Newman (Washington, D.C.)

2587:

Kubilius, J. On a problem in the n-dimensional analytic theory of numbers. Vilniaus Valst. Univ. Mokslo Darbai. Mat. Fiz. Chem. Mokslų Ser. **4** (1955), 5–43. (Lithuanian. Russian summary)

This paper is a sequel to an earlier one by the author [Mat. Sb. (N.S.) **31** (**73**) (1952), 507–542; MR **14**, 847]. It deals with an imaginary quadratic field K over the field of rational numbers, extended, after E. Hecke [Math. Z. **1** (1918), 357–376; **6** (1920), 11–51], to a system of ideal numbers.

Author's summary: "The following theorems are proved. Let $\mathfrak{m} \neq 0$ be an integral ideal of K, $\varphi(\mathfrak{m})$ Euler's function; h the number of ideal classes, g the number of units of the field K; ν an integral ideal number of the field K, $(\nu, \mathfrak{m}) = 1$; φ_1, φ_2 real numbers, $0 < \varphi_2 - \varphi_1 \leqq 2\pi$; x a large positive number. (1) The number of prime ideal numbers $\mathfrak{p}$ of the field K, satisfying the conditions: $\mathfrak{p} \equiv \nu \pmod{\mathfrak{m}}$, $N(\mathfrak{p}) \leqq x$, $\varphi_1 < \operatorname{Arg} \mathfrak{p} \leqq \varphi_2$, is equal to

$$\frac{g(\varphi_2-\varphi_1)x}{2\pi h\varphi(\mathfrak{m}) \ln x}(1+o(1))+O(x^{\vartheta_1}).$$

(2) The number of prime ideal numbers of the field K, satisfying the conditions: $\mathfrak{p} \equiv \nu \pmod{\mathfrak{m}}$, $x < N(\mathfrak{p}) \leqq y$ ($x < y \ll x$), $\varphi_1 < \operatorname{Arg} \mathfrak{p} \leqq \varphi_2$, is equal to

$$\frac{g(\varphi_2-\varphi_1)(y-x)}{2\pi h\varphi(\mathfrak{m}) \ln x}(1+o(1))+O(x^{\vartheta_2}).$$

Here ϑ_1, ϑ_2 are certain absolute constants, $\vartheta_1 < \frac{55}{56}$, $\vartheta_2 < \frac{169}{174}$. (3) Let k be a fixed natural number. For any integral ideal number $\alpha \neq 0$ let $\tau_k(\alpha)$ denote the number of solutions of the equation $\alpha_1 \cdots \alpha_k = \alpha$ in integral ideal numbers $\alpha_1, \cdots, \alpha_k$, where solutions $(\alpha_1', \cdots, \alpha_k')$ and $(\alpha_1'', \cdots, \alpha_k'')$ are considered distinct unless α_j' and α_j'' are associated for each j ($j = 1, \cdots, k$). Then we have the asymptotic formula

$$\sum_{\substack{0<N(\alpha)\leqq x\\ \alpha\equiv\nu \pmod{\mathfrak{m}}\\ \varphi_1<\operatorname{Arg}\alpha\leqq\varphi_2}} \tau_k(\alpha) = \frac{g(\varphi_2-\varphi_1)}{2\pi h\varphi(\mathfrak{m})} R(x, \chi_0, k) + O\{x^{1-1/2k}(\ln x)^{k-1+(k+1)/4k^2}\},$$

where χ_0 is the principal group character mod $\mathfrak{m}$, $R(x, \chi_0, k)$ the residue of the function $x^s s^{-1} Z^k(s, \chi_0)$ at the point $s = 1$, $Z(s, \chi_0)$ the L-function of the field K with character χ_0. In the case $k = 1$ the error term may be replaced by $O(x^{1/2})$.

"All estimates are uniform in x, φ_1, φ_2."

The methods used in this paper seem very similar to those of the author's earlier paper, but a different type of asymptotic formula is discussed. Thus, one of the main topics of the previous paper was an asymptotic formula for the sum

$$\sum_{\mathfrak{p}} \ln N(\mathfrak{p}) \exp\left(-\frac{N(\mathfrak{p}) \ln x}{x}\right)$$

taken over the same range as in (1). For this he now substitutes the 'pure' asymptotic formula (1), in which the summand has been replaced by 1. By a combination of Fourier analysis and contour integration the problem is made to depend on the zeros of certain of Hecke's zeta-functions. The error $o(1)$ arises from a sum $\sum x^{\rho-1}/\rho$, where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of a finite aggregate of these functions and the summation is over a finite range $|\gamma| \leqq T$. This sum is estimated by the 'density' method, i.e., by use of the fact (established in the earlier paper and again, in modified form, in the present one) that zeros with β near to 1 are comparatively scarce. In this way the author is able to estimate this sum as $o(1)$ with a value of T of the order x^δ ($\delta > 0$). The error $O(x^{\vartheta_1})$ then arises from a finite sum of errors $O(x^{1+\epsilon}/T)$ associated with the individual zeta-functions involved. The author notes that it follows from one of the results of his previous paper that, if the Riemann hypothesis is true for the relevant zeta-functions, there exists an infinity of rational primes p expressible in the form $p = a^2 + b^2$ with $|b| < c \ln p$, where c is an absolute constant.

A. E. Ingham (Cambridge, England)

2588:

Kubilyus, I. P. [Kubilius, J. P.] On asymptotic distribution laws of certain number-theoretic functions. Vilniaus Valst. Univ. Mokslo Darbai. Mat. Fiz. Chem. Mokslų Ser. **4** (1955), 45–59. (Russian. Lithuanian summary)

Let $\nu(m)$ be the number of distinct prime factors of the positive integer m, $N\{\cdots\}$ the number of $m \leq x$ satisfying the conditions in $\{\cdots\}$, and

$$G(\omega) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\omega} e^{-u^2/2}du, \quad \Delta(m) = \frac{\nu(m) - \ln_2 x}{\sqrt{(\ln_2 x)}},$$

where $\ln_k$ denotes a k-times repeated logarithm. It is proved that, for fixed distinct integers $a_1, \cdots, a_s \geq 0$, and for a fixed integer $a > 0$, we have

$$(1)\quad x^{-1}N\{\Delta(m+a_1) < \omega_1, \cdots, \Delta(m+a_s) < \omega_s\} = G(\omega_1)\cdots G(\omega_s) + R_1,$$

$$(2)\quad x^{-1}N\{\nu(m) < \nu(m+a) + \omega\sqrt{(2\ln_2 x)}\} = G(\omega) + R_2,$$

where

$$R_1 = O[(\ln_2 x)^{-1/2}(\ln_3 x)^2],$$

and

$$R_2 = O[(\ln_2 x)^{-1/3}(\ln_3 x)^{5/3}],$$

uniformly for all real $\omega_1, \cdots, \omega_s$, and ω, as $x\to\infty$. With $s=1$, $a_1=0$, $R_1=o(1)$, (1) was proved by P. Erdös and M. Kac [Amer. J. Math. **62** (1940), 738–742; MR **2**, 42]; it represents a statistical development of the Hardy-Ramanujan theorem that the normal order of $\nu(m)$ is $\ln_2 m$ [G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Clarendon, Oxford, 1938; Theorem 436]. A different proof was given by H. Delange [C. R. Acad. Sci. Paris **237** (1953), 542–544; MR **15**, 201]. In the same special case W. J. LeVeque improved the error to $R_1 = O[(\ln_2 x)^{-1/4}(\ln_3 x)]$, and conjectured that $R_1 = O[(\ln_2 x)^{-1/2}]$; he also proved (1) with $s=2$, $a_1=0$, $a_2=1$, $R_1=o(1)$, and (2) with $a=1$, $R_2=o(1)$ [Trans. Amer. Math. Soc. **66** (1949), 440–463; MR **11**, 83]. The extensions and refinements of the present paper are obtained by a combination of the sieve method with probability arguments, as in the work of Erdös-Kac and LeVeque, but the details are more complicated owing to the number of variables involved. The probability element enters in connection with the formula

$$e^{-y}\sum_{k<y+\omega\sqrt{y}}\frac{y^k}{k!} = G(\omega) + O\left(\frac{1}{\sqrt{y}}\right) \text{ (uniformly in } \omega),$$

which can, of course, be proved directly. An elaborate generalization of (2) is stated; and analogues of (1) and (2) are proved or stated for some other arithmetical functions. *A. E. Ingham* (Cambridge, England)

2589:

Chandrasekharan, K.; Mandelbrojt, S. On solutions of Riemann's functional equation. Bull. Amer. Math. Soc. **65** (1959), 358–362.

Let λ_n, μ_n $(n \geq 1)$ be positive increasing sequences, and consider solutions of

$$\pi^{-s/2}\Gamma(\tfrac{1}{2}s)\varphi(s) = \pi^{-(\delta-s)/2}\Gamma\{\tfrac{1}{2}(\delta-s)\}\psi(\delta-s)$$

with $\varphi(s) = \sum a_n\lambda_n^{-s}$, $\psi(s) = \sum b_n\mu_n^{-s}$. One result is that if a solution exists when δ is a positive odd integer, and the infima h_λ, h_μ of $\lambda_{n+1}-\lambda_n$, $\mu_{n+1}-\mu_n$ satisfy $h_\lambda\cdot h_\mu = 1$, then $\lambda_{n+1}-\lambda_n = h_\lambda$, $\mu_{n+1}-\mu_n = h_\mu$ for all n. Two other results give conditions under which the existence of a solution implies that $\delta = 1$. One result requires that $h_\mu > 0$ and that $b_n = O(1)$, while the other considers solutions with $\psi(s)$ having the three forms $\sum b_n\mu_n^{-s}$, $\sum d_n\mu_n^{-s}$, $\sum b_nd_n\mu_n^{-s}$ corresponding to three λ-sequences $\{\lambda_n\}$, $\{\lambda_n'\}$, $\{\lambda_n''\}$. The proofs depend on the function $\sum b_n \exp(-2\pi\mu_n s)$. For previous work see, for example, the same authors' paper, C. R. Acad. Sci. Paris **242** (1956), 2793–2796 [MR **18**, 19—where another relevant review appears]. *F. V. Atkinson* (Toronto)

2590:

Schinzel, A. Sur les sommes de trois carrés. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 307–310. (Russian summary, unbound insert)

This paper concerns the problem of the decomposition of a natural integer into a sum of three squares. One of the results proved is the following. A necessary and sufficient condition that a number n admits a decomposition $n = x^2+y^2+z^2$ when $x, y, z > 0$ and $(x, y, z) = 1$ is that n has the two properties (i) $n \not\equiv 0, 4, 7 \pmod 8$ and (ii) n has a divisor of the form $4K-1$, or else is not a "numerus idoneus" [Dickson, *History of the theory of numbers*, Vol. 1, Stechert, New York, 1934; p. 360]. Various corollaries are given, one of which is that "the only natural integers $n < 101{,}200$ with $n \not\equiv 0, 4, 7 \pmod 8$ which are not representable as a sum of three positive squares are 1, 2, 5, 10, 13, 25, 37, 58, 85, 130. The author conjectures that the restriction $n < 101{,}200$ may be dropped. {The subject was considered earlier by Grosswald, Calloway, and Calloway [Proc. Amer. Math. Soc. **10** (1959), 451–455; MR **21** #3376].} *I. A. Barnett* (Cincinnati, Ohio)

2591:

Scourfield, E. J. A generalization of Waring's problem. J. London Math. Soc. **35** (1960), 98–116.

If $\{n_i\}$ be a sequence of integers with $2 \leq n_1 \leq n_2 \leq \cdots$, then it is here proved that for the existence of an integer $r = r(n_j)$ such that all sufficiently large integers N are representable in the form

$$(1)\quad N = x_1^{n_j} + x_2^{n_{j+1}} + \cdots + x_r^{n_{j+r-1}},$$

where the x's are positive integers, it is necessary and sufficient that the infinite series

$$(2)\quad \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots$$

be divergent. In the special case when $\{n_i\}$ is the arithmetic progression $\{n+(i-1)l\}$, with $n \geq 12$, $l \geq 1$, it is shown that the least value of $r(n, l)$ is $\leq C(l)n^{4l+1}(\log n)^{2l}$, $C(l)$ being a certain positive function of l. The restriction $n \geq 12$ has been introduced only for convenience. The necessity of condition (2) is demonstrated in a simple way. For the sufficiency, a variation of the Hardy-Littlewood method is used. *H. Gupta* (Chandigarh)

2592:

Real, Sister M. Anne Cathleen. Waring's problem, modulo p, and the representation symbol. Proc. Iowa Acad. Sci. **66** (1959), 362–364.

Let $p=22k+1$ be a prime. The author sketches a proof that every residue class mod p is the sum of at most four eleventh power residues.
N. G. de Bruijn (Amsterdam)

2593:

Dutta, Mahadev; Debnath, Lokenath. On new partitions of numbers. II. Bull. Calcutta Math. Soc. **51** (1959), 77–78.

[For part I see Dutta, same Bull. **49** (1957), 221–224; MR **20** #5177.] Let ${}_dp(n)$ denote the number of partitions of n in which each part occurs at most d times, so that

$$\sum {}_dp(n)x^n = \prod (1-x^{n(d+1)})(1-x^n)^{-1}.$$

Let $p_d(n)$ be defined by

$$\sum p_d(n)x^n = \prod (1-x^n).$$

The authors note that if $d+1$ is prime then ${}_dp(n)\equiv p_d(n)$ mod $(d+1)$. This fact and some identities and congruences for $p_d(n)$ due to the reviewer enable the authors to deduce a number of congruence properties of ${}_dp(n)$ of which the following is typical: ${}_4p(np+(p^2-1)/6)\equiv 0 \pmod 5$, p prime, $p\equiv 5 \pmod 6$, $(n,p)=1$.
M. Newman (Washington, D.C.)

2594:

Kanagasabapathy, P. On the product $(ax+by+c)(dx+ey)$. Bull. Calcutta Math. Soc. **51** (1959), 1–7.

Let a, b, c, d, e be real numbers satisfying $ad-bc=1$, $c\neq 0$, and suppose that neither of the forms $ax+by$ and $cx+dy$ represents zero for integers x, y not both zero. Using a lemma due to Davenport [Quart. J. Math. Oxford Ser. (2) **3** (1952), 130–138; MR **14**, 252], the author proves that there exist integers x, y not both zero such that

$$|(ax+by+e)(cx+dy)| < (4.2577)^{-1}.$$

This improves slightly an earlier result by the author [ibid. (2) **3** (1952), 197–205; MR **14**, 252].
I. Reiner (Urbana, Ill.)

2595:

Mihaljinec, Mirko. Notes on the E. S. Barnes and H. P. F. Swinnerton-Dyer's paper: The inhomogeneous minima of binary quadratic forms. III. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II **14** (1959), 121–134. (Serbo-Croatian summary)

[The paper of the title is Acta Math. **92** (1954), 199–234; MR **16**, 802.] Let $\mathscr{L}$ denote a two-dimensional vector space of points (ξ, η) with determinant Δ. A lattice $\mathscr{L}$ is said to be R_m-admissible if for the region $R_m: -1<\xi\eta<m$, with $m\geqq 1$ and real, there is no point of $\mathscr{L}$ contained in R_m. The lattice $\mathscr{L}_2$ is a multiple of lattice $\mathscr{L}_1$, if there are real numbers $C\geqq 1$ and k such that the transformation $f(\xi,\eta)=(C\xi/k, Ck\eta)$ transforms the points of $\mathscr{L}_1$ onto points of $\mathscr{L}_2$.

The "minimal determinants" $D_m^{(k)}$, for $k=1, 2, 3, \cdots$, are defined inductively as follows: $D_m^{(k)}=\inf \Delta(\mathscr{L})$, where $\mathscr{L}$ runs over all R_m-admissible lattices which are not multiples of any R_m-admissible lattice with determinant $\leqq D_m^{(k-1)}$ and $\Delta(\mathscr{L})>D_m^{(k-1)}$ if $k\geqq 2$, and without restriction if $k=1$.

Barnes and Swinnerton-Dyer have determined $D_m^{(1)}$ for m between $1.9090\cdots$ and $2.1251\cdots$ and Davenport has shown that $D_2^{(1)}$ is isolated. The author determines the values of $D_m^{(k)}$ for $m=2001/1000$ and $k=1, 2, 3, 4$. The first three of these minima are isolated. The minima for $k=4$ and $k=5$ are equal. Associated results are also obtained.
B. W. Jones (Boulder, Colorado)

2596:

Rogers, K.; Straus, E. G. A class of geometric lattices. Bull. Amer. Math. Soc. **66** (1960), 118–123.

Hajós's theorem [Math. Z. **47** (1941), 427–467; MR **3**, 302] gives a full enumeration of the critical lattices Λ of the cube $K: |x_1|\leqq 1, |x_2|\leqq 1, \cdots, |x_n|\leqq 1$ in R^n: Let A be an $n\times n$ matrix the columns of which form a basis of Λ. Then Λ is critical if the basis can be chosen such that A has 1's in the diagonal and 0's above it. A is said to have the property P if for every vector $\mathbf{u}\neq\mathbf{0}$ with integral components at least one component of $A\mathbf{u}$ is an integer not 0. If it were true that (i) "every A with property P has an integral row", Hajós's theorem would follow from a result by Siegel stating that every point $\neq\mathbf{0}$ of a critical lattice of K has at least one integral coordinate $\neq 0$; but (i) is false for $n\geqq 5$. The authors investigate in detail matrices A with the property P and prove in particular the following theorems. (1) Let k be an algebraic number field of class number 1; let j be the ring of integers in k; and let A be an $n\times n$ matrix with elements in k. If for every vector $\mathbf{u}\neq\mathbf{0}$ with components in j at least one component of $A\mathbf{u}$ is in j and is $\neq 0$, then det A is likewise in j and $\neq 0$. (2) If A is as in (1) and $0<|\text{norm}(\det A)|<1$, there exists a $\mathbf{u}\neq\mathbf{0}$ with components in j such that $A\mathbf{u}$ has no component $\neq 0$ in j.
K. Mahler (Manchester)

2597:

Brun, Viggo. Algorithmes euclidiens pour trois et quatre nombres. Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 45–64. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

The author points out that Euclid's algorithm for 2 numbers $a>b$ may be considered to be based on subtraction as well as on division. He remarks that Jacobi's algorithm for 3 numbers is of no great practical use, exactly because of the fact that it is based on division. The author's algorithm is based on subtraction [cf. V. Brun, Norske Vid. Selsk. Skr. Mat. Nat. Kl. **1919**, no. 6, **1920**, no. 6] and related to, although different from, work of Pipping [Acta Acad. Abo. **21** (1957), no. 1; MR **18**, 565] and David [Publ. Sci. Univ. Alger. Sér. A **3** (1956), 1–102; MR **20** #7012]. It is explained without use of geometrical methods: Start with the triple $a_0\geqq b_0\geqq c_0$ and form the triple a_0-b_0, b_0, c_0, which in ordered form may be written as $a_1\geqq b_1\geqq c_1$. Repeating this process, one gets a sequence of triples $a_n\geqq b_n\geqq c_n$ $(n\geqq 0)$ to each of which a matrix

$$M_n = \begin{pmatrix} x_n' & y_n' & z_n' \\ x_n'' & y_n'' & z_n'' \\ x_n''' & y_n''' & z_n''' \end{pmatrix}$$

with integer elements X, Y, Z may be constructed, starting with

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If M_n has been constructed already, form

$$M_n^* = \begin{pmatrix} x_n' & y_n' & z_n' \\ x_n'+x_n'' & y_n'+y_n'' & z_n'+z_n'' \\ x_n''' & y_n''' & z_n''' \end{pmatrix}$$

and permute the rows of M_n^* in exactly the same way in which the triple a_n-b_n, b_n, c_n must be permuted to get the triple a_{n+1}, b_{n+1}, c_{n+1}. The result is the matrix M_{n+1}. Now for each $n\geqq 0$ three possibilities may occur, called α, β, γ; viz.: (α) $a_n-b_n\geqq b_n$; (β) $b_n>a_n-b_n\geqq c_n$; (γ) $c_n>a_n-b_n$. If we now put

$$\xi_n = a_nY-b_nX = a_nX(Y/X-b_n/a_n),$$

to each triple a_n, b_n, c_n a triple ξ_n', ξ_n'', ξ_n''' corresponds by applying the above definition of ξ_n for each of the three rows (X, Y, Z) of M_n. Finally put $\omega_n=\text{Max}(|\xi_n'|, |\xi_n''|, |\xi_n'''|)$. Now several properties can be proved, e.g., that always $\omega_1\geqq\omega_2\geqq\omega_3\geqq\cdots$; that when infinitely many cases γ occur the algorithm converges, etc. Similar results hold for $m\geqq 4$ numbers. Examples for $m=3, 4$ are given. *J. F. Koksma* (Amsterdam)

2598:

Gelfond, A. O. ★Transcendental and algebraic numbers. Translated from the first Russian edition by Leo F. Boron. Dover Publications, Inc., New York, 1960. vii+190 pp. $1.75.

Translation of the 1952 edition [GITTL, Moscow; MR 15, 292].

2599:

Melzak, Z. A. A countable interpolation problem. Proc. Amer. Math. Soc. **11** (1960), 304–306.

Let $\mathscr{X}$ be the metric space of all order-preserving homeomorphisms f of $I=[0, 1]$ onto itself, the metric being $\rho(f_1, f_2)=\max_I|f_1(x)-f_2(x)|$. Let $\mathscr{X}_\alpha$ ($\alpha>2$) be the subset of $\mathscr{X}$ consisting of all functions $f\in\mathscr{X}$ whose values, for algebraic $x\in I$, are either rational or transcendental and approximable to degree $>\alpha$. (I.e., the inequality $|x-p/q|<q^{-d}$ has infinitely many rational integral solutions p, q, for some $d>\alpha$.) Then it is shown that $\mathscr{X}_\alpha$ is a dense G_δ-set of second category in $\mathscr{X}$. The theorem remains true if "approximable to degree $>\alpha$" is replaced by "a Liouville number".

W. J. LeVeque (Ann Arbor, Mich.)

FIELDS

See also 2611.

2600:

Cunningham, F., Jr. A construction of the rational numbers. Amer. Math. Monthly **66** (1959), 769–777.

The author defines a Peano succession as a pair (N, s), where N is a set, $s: N\to N$ a function such that s^{-1} is also a function, $s(N)$ is not equal to N, and there is an element e of N such that N is the only tail containing e (a tail is a subsuccession formed by restricting s to a subset of N closed under s). The author then shows that from a Peano succession one can construct a free cyclic group. The development leads to several delightful lemmas such as "If A and B are non-empty tails, then $A\cap B$ is a non-empty tail". Finally, from the free cyclic group obtained, the author shows that there exists a prime field R of characteristic zero such that the free cyclic group is isomorphic to the integer group of R. {It is doubtful if this rather abstruse approach will appeal to elementary students.} *R. G. Stanton* (Waterloo, Ont.)

2601:

Bialynicki-Birula, A.; Browkin, J.; Schinzel, A. On the representation of fields as finite unions of subfields. Colloq. Math. **7** (1959), 31–32.

This article contains a short proof of the fact that no field can be represented as a finite union of proper subfields. A counterexample is exhibited to the corresponding assertion for integral domains.

S. W. Golomb (Pasadena, Calif.)

2602:

Swift, J. D. Construction of Galois fields of characteristic two and irreducible polynomials. Math. Comput. **14** (1960), 99–103.

If 2 is a primitive root of the odd prime p, then $(x^p+1)/(x+1)=f(x)$ is irreducible over GF(2) and the residues mod $(2, f(x))$ determine $\mathrm{GF}(2^{p-1})$ [see Albert, *Fundamental concepts of higher algebra*, Univ. of Chicago Press, Chicago, Ill., 1958; MR **20** #5190]. The author finds (primitive) polynomials g whose powers generate the nonzero elements of $\mathrm{GF}(2^{p-1})$, and he notes that with $m=(2^{p-1}-1)/(2^d-1)$ for $d|(p-1)$, g^m generates $\mathrm{GF}(2^d)$. The author believes that all $\mathrm{GF}(2^d)$ can be generated starting with a suitable p. In fact, using trial and error, he calculates g and g^m for all $p<144$ and finds that the unattained d turn out to be 8, 16, 17, $\cdots$.

The irreducible polynomials of degree d over GF(2) can be given by the dependence polynomials $f(z)=0$ of the primitive z of $\mathrm{GF}(2^d)$. The author gives a simple method of constructing such polynomials, making fullest use of the binary structure of the SWAC. *H. Cohn* (Tucson, Ariz.)

2603:

Carlitz, L. A characterization of algebraic number fields with class number two. Proc. Amer. Math. Soc. **11** (1960), 391–392.

Let h denote the class number of an algebraic number field Z over the rational numbers. A necessary and sufficient condition for $h=1$ is that unique factorization into primes holds for the integers of Z. The author shows that a necessary and sufficient condition for $h\leqq 2$ is that for every nonzero integer α in Z the number of primes π_j in every factorization $\alpha=\pi_1\pi_2\cdots\pi_k$ depends only on α.

T. M. Apostol (Pasadena, Calif.)

2604:

Billevič, K. K. On the identity of two algebraic fields of order n. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 571–574. (Russian)

Gegeben seien zwei algebraische Zahlkörper Ω und Ω_1 durch erzeugende Gleichungen $f(x)=0$ und $f_1(x)=0$, mit ganzrationalen Koeffizienten und höchstem Koeffizienten 1. Verf. stellt sich die Aufgabe, ein Verfahren anzugeben, das die eventuelle Gleichheit der Körper Ω und Ω_1 festzustellen gestattet. Dabei wird, wie es scheint, auf die

praktische Durchführbarkeit der auftretenden Rechnungen Wert gelegt. Das Verfahren verlangt u.a. die Aufstellung von Ganzheitsbasen $\omega_1, \cdots, \omega_n$ bezw. $\tau_1, \cdots, \tau_n$ der beiden Körper; es beruht dann im Wesentlichen auf den Minkowskischen Gitterpunktsätzen, angewandt in geeigneter Weise auf die durch die ω_i bezw. τ_j erzeugten Gitter im n-dimensionalen Raum.

P. Roquette (Tübingen)

2605:

Nobusawa, Nobuo. On integral basis of algebraic function fields with several variables. Osaka Math. J. **11** (1959), 63–69.

Es sei K ein algebraischer Funktionenkörper in 2 Variablen, und w eine diskrete, zweirangige Bewertung von K, mit zugehörigem Bewertungsring $\mathfrak{o}_0$. Es sei ferner L eine endliche Erweiterung von K und $\mathfrak{o}$ die ganzabgeschlossene Hülle von $\mathfrak{o}_0$ in L; diese ist gleich dem Durchschnitt der Bewertungsringe $\mathfrak{o}_i$ von L, die zu den verschiedenen Fortsetzungen w_i von w auf L gehören. Verf. setzt sich die Aufgabe, die Struktur von $\mathfrak{o}$ als $\mathfrak{o}_0$-Modul zu bestimmen. Das Hauptergebnis lautet folgendermaßen: $\mathfrak{o}$ ist eine direkte Summe einer gewissen Anzahl n_0 von einrangigen $\mathfrak{o}_0$-Moduln, und von $n-n_0$ unendlichrangigen $\mathfrak{o}_0$-Moduln, mit $n=(L:K)$. Dabei wird die Anzahl n_0 der Summanden vom Range 1 wie folgt bestimmt: Man identifiziere die Wertgruppe von w mit der lexikographisch geordneten Gruppe der Paare (α_1, α_2) ganzrationaler Zahlen, wobei die lexikographische Ordnung etwa von rechts nach links zu lesen ist. Entsprechend normiere man w_i. Dann sei $(e_1^{(i)}, e_2^{(i)})$ die Verzweigungszahl von w_i über w, d.h. $w_i(a)=(e_1^{(i)}, e_2^{(i)})\cdot w(a)$ für $a\in K$. (Das Produkt $e=e_1^{(i)}e_2^{(i)}$ ist dann die Verzweigungsordnung im gewöhnlichen Sinne von w_i über w.) Mit diesen Bezeichnungen ist nun $n_0=\sum_i e_1^{(i)}f_i$; hier bedeutet f_i, wie üblich, den Restklassengrad von w_i über w.—Genau dann besitzt L/K eine Ganzheitsbasis bezüglich w, wenn $\mathfrak{o}$ keine Summanden unendlichen Ranges besitzt, d.h. wenn alle $e_2^{(i)}=1$. Der Verf. teilt mit, daß dieses Resultat sinngemäß auf den Fall eines Funktionenkörpers in r Variablen $(r\geq 2)$ verallgemeinert werden kann, wobei man dann eine r-rangige diskrete Bewertung zu untersuchen hat, oder allgemeiner eine solche Bewertung, für welche Rang plus Dimension gleich r ist.

P. Roquette (Tübingen)

2606:

Капланский, И. [Kaplansky, Irving.] ★Введение в дифференциальную алгебру. [An introduction to differential algebra.] Translated from the English by G. I. Kleinerman; edited by M. M. Postnikov. Biblioteka Sbornika "Matematika". Izdat. Inostr. Lit., Moscow, 1959. 85 pp. 2.85 r.

Translation of Actualités Sci. Ind. No. 1251, Hermann, Paris, 1957 [MR **20** #177].

ALGEBRAIC GEOMETRY

See also 3011, 3018.

2607:

Gallarati, Dionisio. Una superficie dell'ottavo ordine con 160 nodi. Atti Accad. Ligure **14** (1957), 44–50 (1958). (English summary)

Author's summary: "Construction of an algebraic F^8 possessing 160 isolated double points, as branch surface of a double S_3."

2608a:

Aucoin, C. V.; Perry, N. C. On the relation of perfect points to a Fibonacci sequence. Univ. Nac. Tucumán. Rev. Ser. A **12**, 7–10 (1959).

2608b:

Morelock, J. C.; Perry, N. C.; Hutcherson, W. R. Fibonacci sequence applied to quadratic transformations. Univ. Nac. Tucumán. Rev. Ser. A **12**, 81–84 (1959).

These two papers contain essentially the same algebraic results applied in two geometric contexts. A typical algebraic result can be expressed in the following form. If

$$A=\begin{bmatrix}0&0&2\\0&1&1\\1&1&0\end{bmatrix},\quad \mathbf{v}=[0, 2, 3],\quad A^r\mathbf{v}=[w_r, x_r, y_r]$$

then $|y_r-w_r|$ form a Fibonacci sequence.

T. G. Room (Sydney)

2609:

Gaeta Maurelo, Federico. Teoria geometrico-tensoriale dei complessi di sottospazi S_d di S_n. Confer. Sem. Mat. Univ. Bari **41–42**, 36 pp. (1958).

Si tratta di un'esposizione elementare e riassuntiva della teoria dei complessi di spazi lineari, preliminare ad alcuni risultati recentemente ottenuti dall'A. nella teoria dell'eliminazione e sul teorema di Bézout [Gaeta, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 148–150; **23** (1957), 389–394; **24** (1958), 269–276; MR **17**, 788; **21** #49; **20** #7027].

F. Gherardelli (Florence)

2610:

Knight, A. J. Algebraic sub[illegible]ieties of an abelian variety. Proc. London Math. S[illegible]. (3) **10** (1960), 253–280.

The author considers the free groups generated by those (algebraic) cycles of a p-dimensional abelian variety (over the complex field) which are complete intersections of $(p-1)$-dimensional irreducible subvarieties, and proceeds to find the number of generators of such groups, modulo homology on the torus. The methods are largely elementary and computational; in dealing with large arrays of superscripts and subscripts, the author has adopted a notation which makes things easy for the printer, but hard on the reader.

I. Barsotti (Providence, R.I.)

2611:

Шевалле, Клод [Chevalley, Claude]. ★Введение в теорию алгебраических функций от одной переменной. [Introduction to the theory of algebraic functions of one variable.] Translated from the English by Z. I. Borevič. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959. 334 pp. 11.20 r.

Translation of 1951 edition [American Mathematical Society, New York; MR **13**, 64].

2612:

Hasse, Helmut. The Riemann hypothesis for function fields over finite fields of constants. Univ. Madrid. Publ. Sec. Mat. Fac. Ci. **1**, no. 2 (1957), 142 pp. (Spanish)

Ce travail est un cours, fait à l'Université de Madrid, où l'auteur expose une démonstration de l'hypothèse de Riemann dans le corps des fonctions algébriques d'une variable sur un corps fini [on sait que la première démonstration générale de cette hypothèse (qu'on va écrire, par abréviation, "F-hypothèse de Riemann") est due à A. Weil; voir A. Weil, *Foundations of algebraic geometry*, Amer. Math. Soc., New York, 1946; *Sur les courbes algébriques et les variétés qui s'en déduisent*; Hermann, Paris, 1948; MR **9**, 303; **10**, 262]. On sait que la démonstration de ce théorème se réduit, d'une manière classique aujourd'hui, à la démonstration de la propriété suivante de "positivité" de la trace (au sens de Weil) $\sigma(\xi)$ de classe ξ de correspondances: si ξ' est la classe transposée de $\xi \neq 0$, on a $\sigma(\xi\xi') > 0$. Il s'agit donc, finalement, d'un théorème de la théorie des correspondances, et Weil développe cette théorie à partir de sa théorie des intersections, exposée dans ses *Foundations of algebraic geometry*. Mais, de cette manière, on ne peut parvenir à la démonstration de la F-hypothèse de Riemann sans avoir assimilé, au préalable, les 250 pages des "Foundations", et on sait combien est malaisée (malgré la profondeur de sa conception et sa cohérence interne) la lecture de ce livre. La démonstration de l'auteur (très différente de celle de Weil) exige moins de connaissances préalables de géométrie algébrique et est plus accessible que celle de Weil pour ceux qui ne s'occupent pas spécialement de ce domaine. L'auteur, dans son exposé, fonde la théorie des correspondances sur celle des "restes de diviseurs" (due aux recherches de Deuring, Roquette, etc., et qu'il serait trop long d'exposer ici), au lieu de la fonder sur celle (plus compliquée, mais de portée plus grande) des intersections, ce qui lui permet d'arriver à la démonstration de la F-hypothèse de Riemann au bout de 140 pages seulement. Mais ce raccourci ne va pas sans compensations, car dans la méthode de Weil, d'inspiration plus géométrique, on sent mieux que dans celle de Hasse les raisons profondes des résultats et des notions employées.

Remarques du référent: L'auteur emploie (et, semble-t-il, délibérément) une même notation, celle de produit, pour désigner 3 opérations différentes: le composé d'un diviseur (considéré comme un homomorphisme du corps dans celui des constantes) avec une application, le transformé d'un diviseur par un isomorphisme et le reste d'un diviseur par rapport à un autre diviseur. Malgré ce que le contexte permet toujours de décider, après réflexion, de quelle opération il s'agit dans chaque cas particulier, cette manière de faire impose au lecteur un travail de vérification à mon avis superflu et peut désorienter un lecteur inexpérimenté.

D'autre part, le travail comporte un certain nombre d'errata, dont certains sont très mal placés et peuvent empêcher la compréhension des définitions. En voici quelques-uns: p. 28, lignes 15 et 18: lire $\mathrm{gr}_{\tilde{K}'/\Omega}(\mu)$ au lieu de $\mathrm{gr}_{\tilde{K}'/\Delta}(\mu)$; p. 57, lignes 13, 15: la lettre n joue, dans le texte, deux rôles différents, ce qui rend les formules incompréhensibles; il faudrait, par exemple, les écrire "$\Delta n = \Omega(\theta)$ donde $t\nu = \theta^3$, $t'\nu' = \theta^2$" et "$\bar{v}$ numerador de θ"; p. 71, ligne 21: lire Δ/Ω au lieu de K/Ω; p. 76, ligne 29: lire $v'\mu'$ au lieu de $v\mu'$; p. 82, lignes 14, 25: lire $D_{\mu,\nu}$ au lieu de $D_{v,\nu}$; p. 87, ligne 9: lire "$(\underline{\mu}, \underline{\nu})$-resultante" au lieu de "$(\mu, \underline{\nu})$-resultante"; p. 105, ligne 6: lire $\mathrm{gr}_{\tilde{K}'/\Omega}(\underline{\mu}) = \mathrm{gr}_{\tilde{\Delta}/K}(\mathfrak{a}) = m'$ au lieu de $\mathrm{gr}_{\tilde{K}'/\Omega}(\underline{\mu})\mathrm{gr}_{\tilde{\Delta}/K}(\mathfrak{a}) = m'$; p. 125, ligne 18: lire "numerador" au lieu de "denominador".

M. Krasner (Paris)

2613:

Endo, Shizuo. On ideals defining non-singular algebraic varieties. Proc. Japan Acad. **36** (1960), 3–5.

If R is a regular ring of dimension d and $\mathfrak{p}$ is a prime ideal in R of rank s such that $R/\mathfrak{p}$ is also a regular ring, it is shown that the number of elements in a minimal base of $\mathfrak{p}$ is not greater than $s(d-s+1)+1$. Hence in particular if V is an irreducible non-singular variety of dimension r in an affine space $A_n(k)$, then the ideal defining V over the field k is generated by at most $(n-r)(r+1)+1$ elements.

H. T. Muhly (Iowa City, Iowa)

2614:

Muhly, H. T. Complete ideals in local rings. Proc. Amer. Math. Soc. **11** (1960), 361–367.

Let P be a point of an arithmetically normal surface V which is defined over an algebraically closed ground-field. A birational transform of V is constructed by first applying to V a quadratic transformation with centre P, and then passing to the derived arithmetically normal surface U_δ which belongs to the character of homogeneity δ. The author gives a relation between the Hilbert functions of the surfaces V and U_δ.

Let L be a noetherian local ring with maximal ideal M, and let Q_n denote the integral closure (or v-completion) of M^n. The author begins with a number of interesting algebraic results of which the most significant two are: (a) if L is an integrally closed domain, then $\bigcap_{n=1}^{\infty} Q_n = (0)$; (b) if L is also 2-dimensional and satisfies certain finiteness conditions, then the length $\lambda(Q_n)$ of Q_n is equal to a polynomial of degree 2 in n, for n sufficiently large.

With the geometric situation described above, if L is the local ring of P on V, then the conditions referred to in (b) are satisfied, and the required relation between the Hilbert functions χ is $\chi(V, 2n\delta) - \chi(U_\delta, n) = \lambda(Q_n)$. In particular the constant term in $\lambda(Q_n)$ is the difference in the arithmetic genera of V and U_δ.

D. Kirby (Leeds)

2615:

Nishimura, Toshio. On the equivalence theory of the extended Grassmann variety. Bull. Kyoto Gakugei Univ. Ser. B, no. 5 (1954), 24–28.

L^r und L^s $(r<s)$ seien Unterräume des projektiven L^N mit $L^r \subset L^s$. Die Gesamtheit aller derartigen voll inzidenten Raumpaare (L^r, L^s) nennt der Verf. $H(r, s; N)$. Man kann diese Gesamtheit ersichtlich auf eine bestimmte Untermenge des Segreproduktes $G_{N,r} \times G_{N,s}$ der Graßmannschen $G_{N,r}$ und $G_{N,s}$ abbilden. Verf. zeigt dann folgendes. (a) $H(r, s; N)$ ist über dem Grundkörper k algebraisch. (b) Läßt man die L^r und L^s je gewiße Schubertbedingungen erfüllen, so ergibt sich eine Teilmenge von $H(r, s; N)$. Diese Teilmengen bilden eine Basis auf $H(r, s; N)$. (c) Die verschiedenen Arten von Äquivalenz auf $H(r, s; N)$ (numerisch, algebraisch und linear) fallen zusammen.

W. Burau (Hamburg)

LINEAR ALGEBRA

See also 2578, 2900, 2904, 2906, 2928, B3438.

2616:

Valentiner, Siegfried. ★Vektoren und Matrizen. 2. Aufl. (9., erweiterte Aufl. der "Vektoranalysis"). Mit einem Anhang: Aufgaben zur Vektorrechnung, von Hermann König. Sammlung Göschen Bd. 354/354a. Walter de Gruyter & Co., Berlin, 1960. 202 pp. DM 5.80.

The booklet has three parts: Rechnungsregeln der Vektoranalysis; Anwendung in einigen physikalischen Gebieten; lineare Vektorfunktionen, Matrizen, Dyaden. There is a supplement with 42 exercises (with solutions), and a collection of 50 important formulas.

2617:

Bogdanov, Yu. S. Lyapunov norms in linear spaces. Mat. Sb. (N.S.) **49 (91)** (1959), 225–231. (Russian)

By Lyapunov norm [sic] V in a linear space A is meant a function $V: A \rightarrow \Delta$ (a well-ordered set) for which $V(a+a') \leqq \max(V(a), V(a'))$ and $V(\lambda a) = V(a)$ for $\lambda \neq 0$. The properties of linear spaces with such valuations are studied. For example, a basis $\{a_\alpha\}$ for A is called a V-basis if, for any finite linear combination $a = \sum \lambda_\alpha a_\alpha$, one has $V(a) = \max\{V(a_\alpha): \lambda_\alpha \neq 0\}$. Any ordinary basis can be simply deformed into a V-basis, in the finite dimensional case. Changes of V-bases are studied. Linear transformations T for which $V(a) = V(T^{-1}a) = V(Ta)$ are called Lyapunov similarities. They send V-bases into V-bases. There are further results of this character, and there are quite a few examples. *R. Arens* (Los Angeles, Calif.)

2618:

Vilenkin, N. Ya. On an estimate of the largest eigenvalue of a matrix. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 55–57. (Russian)

For the arbitrary square matrix $\|a_{ik}\|^n$, arrange the sums $S_i = \sum_{k=1}^n |a_{ik}|$ $(i = 1, \cdots, n)$ in order of decreasing magnitude: $S_{m_1} \geqq S_{m_2} \geqq \cdots \geqq S_{m_n}$. Setting $j = m_1$, introduce the additional notation $T_{ik} = S_i - |a_{ik}|$ $(i, k = 1, \cdots, n)$ and

$$U_r = \tfrac{1}{2}T_{rj} + |a_{jj}| + \sqrt{((T_{rj} - |a_{jj}|)^2 + 4T_{jj}|a_{rj}|)} \quad \text{for } r \neq j,$$
$$= S_{m_2} \quad \text{for } r = j$$

$(r = 1, \cdots, n)$. In this article the following bound is established for the moduli of the eigenvalues λ of matrix $\|a_{ik}\|$: $|\lambda| \leqq \max_r U_r$. {Reviewer's note: In the article there is a misprint in the principal formula (a_{rj} in place of $|a_{rj}|$).} *F. R. Gantmaher* (RŽMat. **1959** #8826)

2619:

Richter, Hans. Zur Abschätzung von Matrizennormen. Math. Nachr. **18** (1958), 178–187.

Let A be an n by n matrix, $N(A)$ its Frobenius norm, $\det A \neq 0$. Let $q(A) \geqq 1$ be the solution of

$$N^2(A)/n(|\det A|^2)^{1/n} = (n-1)/nq(A) + q^{n-1}(A)/n.$$

Then $1 \leqq q(AB) \leqq q(A)q(B)$. For given $N(A)$, $N(B)$, $\det A$ and $\det B$, the upper bound is sharp; the lower bound is obtained if and only if AB is a multiple of a unitary matrix.

An inequality in a similar spirit compares measures of A with measures of A^{-1}. *A. J. Hoffman* (New York)

2620:

den Broeder, George G., Jr.; Smith, Harry J. A property of semi-definite hermitian matrices. J. Assoc. Comput. Mach. **5** (1958), 244–245.

Let $A = S + iQ$ be semi-definite hermitian, $S' = S$, $Q' = -Q$ two real matrices. By showing that $Sz = 0$ implies $Qz = 0$ and $Az = 0$, it is proved that $\operatorname{rank} S \geqq \operatorname{rank} Q$ and $\operatorname{rank} S \geqq \operatorname{rank} A$. (The example $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ shows that $\operatorname{rank} S$ can be greater than $\operatorname{rank} A$.) Thus one can compute the real and imaginary part of $A^{-1} = X + iY$ by solving $SX - QY = I$, $SY + QX = 0$; indeed if A^{-1} exists, then so does S^{-1}. *H. Schwerdtfeger* (Montreal)

2621:

Wilf, Herbert S. Almost diagonal matrices. Amer. Math. Monthly **67** (1960), 431–434.

By an "almost diagonal matrix" the author means one which differs from a diagonal matrix by a matrix of rank 1. The characteristic polynomial is exhibited explicitly; conditions are given for the roots to be distinct; and for this case the matrix P of appropriately normalized characteristic vectors, and P^{-1}, are given explicitly. Only the real case is considered, but the results appear to extend to the complex case with obvious modifications. A final byproduct is a condition for definiteness when the matrix is symmetric.

A. S. Householder (Oak Ridge, Tenn.)

2622:

Gregory, Robert T. Defective and derogatory matrices. SIAM Rev. **2** (1960), 134–139.

The concepts are defined and contrasted, with examples. *A. S. Householder* (Oak Ridge, Tenn.)

2623:

Glidman, S. Some theorems on the Gauss algorithm and the matrix orthogonalization process. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 373–379. (Russian summary, unbound insert)

Let $\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n$ denote n vectors of a vector space. The author defines external vectors $\bar{a}_{r,s}$ inductively as follows: $\bar{a}_{1,s} = \bar{a}_1 \times \bar{a}_{s+1}$; $\bar{a}_{k,s} = \bar{a}_{k-1,1} \times \bar{a}_{k-1,s+1}$. He defines another set of vectors $\bar{w}_{r,s}$ and shows them isometric to the external vectors defined above. He shows that the vectors $\bar{w}_{r,s}$ and an allied set $\bar{u}_{r,s}$ may be expressed in terms of the vectors $\bar{a}_i$ by means of symbolic determinants. Application is made to the orthogonalization of a given matrix. The details are notationally too complex to exhibit here. *B. W. Jones* (Boulder, Colorado)

2624:

Annin, B. D. The Lagrange-Sylvester formula for tensor functions, depending on two tensors. Dokl. Akad. Nauk SSSR **133** (1960), 743–744 (Russian); translated as Soviet Math. Dokl. **1**, 899–900.

Given are, in three dimensions, the symmetric bivalent tensors H, T_1 and T_2; H is a tensor function of T_1 and T_2. Then H can be written

$$H = K_1G+K_2T_1+K_3T_1^2+K_4T_2 + K_5(T_1T_2+T_2T_1)+K_6(T_1^2T_2+T_2T_1^2);$$

where $G=\|\delta_{ij}\|$ is the unit tensor. Written with respect to the principal directions of T_1, the equation for H is equivalent to the six equations (do not sum on i or j):

$$H_{ij} = K_1\delta_{ij}+K_2\delta_{ij}\lambda_j+K_3\delta_{ij}\lambda_j^2+K_4T_{ij} + K_5T_{ij}(\lambda_i+\lambda_j)+K_6T_{ij}(\lambda_i^2+\lambda_j^2), \quad i, j = 1, 2, 3;$$

here $H=\|H_{ij}\|$, $T_1=\|\delta_{ij}\lambda_j\|$, $T_2=\|T_{ij}\|$; the characteristic numbers λ_i are taken as different from each other. From these six equations $K_1, K_2, \cdots, K_6$ can be found; by the substitution of these values into the expression for H the required formula is obtained and written out explicitly.

D. J. Struik (Cambridge, Mass.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 2614.

2625:

Dickson, Leonard Eugene. ★Algebras and their arithmetics. Dover Publications, Inc., New York, 1960. xii+241 pp. Paperbound: $1.35.

Unaltered republication of first edition [Univ. of Chicago Press, Chicago, Ill., 1923].

2626:

Posner, Edward C. Prime rings satisfying a polynomial identity. Proc. Amer. Math. Soc. **11** (1960), 180–183.

A prime associative ring satisfies a polynomial identity if and only if it has a two-sided ring of quotients which is a ring of all $r\times r$ matrices over a finite-dimensional central division algebra. This extends a result of the reviewer [J. London Math. Soc. **30** (1955), 464–470; MR **17**, 122] for rings without zero divisors. The proof is based on the fact that these rings satisfy Goldie's conditions for the existence of rings of quotients which are $r\times r$ matrix rings over division rings [Proc. London Math. Soc. (3) **8** (1958), 589–608; MR **21** #1988]. A corollary: Algebras with polynomial identities, for which every element is a sum of nilpotent elements, are nil algebras.

S. A. Amitsur (Jerusalem)

2627:

Goldie, A. W. Semi-prime rings with maximum condition. Proc. London Math. Soc. (3) **10** (1960), 201–220.

Previous results of the author [cf. #2626 for reference] are extended to semi-prime rings. The semi-prime ring R is assumed to satisfy the following conditions: (1) R has finite dimension as a right R-module, and (2) the ascending chain condition holds for the set of annihilator right ideals. It is proved that R has a right quotient ring $Q=\{ab^{-1}|\ a, b\in R, b$ regular$\}$ which is semi-simple and satisfies the minimum condition for right ideals. Conversely, if Q is a semi-simple ring with minimum condition and if R is a subring of Q such that $Q=\{ab^{-1}|\ a, b\in R, b$ regular$\}$ then R is semi-prime satisfying (1) and (2). Conditions are given under which Q is also a left quotient ring of R. These results are used to prove the following theorem concerning a weakly closed system S of an algebra A over a commutative ring. If A satisfies the maximum condition for right ideals and if each element of S is nilpotent, then it is proved that the enveloping algebra S^* of S is nilpotent. This theorem was proved by Jacobson [*Structure of rings*, Amer. Math. Soc., Providence, R.I., 1956; MR **18**, 373; p. 201] in case A satisfies the minimum condition for right ideals.

R. E. Johnson (Rochester, N.Y.)

2628:

Tominaga, Hisao. A note on conjugates. II. Math. J. Okayama Univ. **9** (1959/60), 1–3.

[For part I see same J. **7** (1957), 75–76; MR **21** #1981.] In this note the author effortlessly generalizes to primary rings results of I. Herstein, F. Kasch, W. R. Scott, and himself on conjugates in division rings, and simple rings. {Note: $\#(\bar{R})$ should replace $\#(R)$ in the statements of theorems 1 and 1'.}

Carl Faith (Princeton, N.J.)

2629:

Sakuma, Motoyoshi. On the theory of multiplicities in finite modules over semi-local rings. J. Sci. Hiroshima Univ. Ser. A **23**, 1–17 (1959).

The known theory of multiplicity in local rings or in finite modules over local rings is generalized in this article to the case of finite modules over semi-local rings. At the end of the article, the author proves that the complete tensor product over a field is an exact functor in a certain sense.

M. Nagata (Kyoto)

2630:

Endler, Otto. Modules and rings of fractions. Summa Brasil. Math. **4**, 149–182 (1959).

Soient A un anneau (non nécéssairement commutatif), M un A-module à droite, et S un monoide commutatif d'endomorphismes de M. Généralisant la méthode classique des classes d'équivalence de couples $(m, s)\in M\times S$, on définit un module de fractions M_S. Étude de l'homomorphisme $M\to M_S$; variations de M_S avec S. Lorsque $M=A$ et que les éléments de S sont aussi des endomorphismes pour la structure de A-module à gauche de A (ce qui implique, si A admet un élément unité, que tout élément de S est une homothétie centrale), M_S est un anneau. Dans le cas commutatif, cette généralisation permet l'étude des anneaux de fractions d'anneaux sans élément unité. Propriétés fonctorielles. Comparaison avec les anneaux et modules de fractions classiques.

P. Samuel (Clermont-Ferrand)

2631:

Nagahara, Takasi; Tominaga, Hisao. Some remarks on Galois extensions of division rings. Math. J. Okayama Univ. **9** (1959/60), 5–8.

Soient K un corps non commutatif, L un sous-corps de K tel que K soit une extension galoisienne de L (non nécessairement finie), K' un corps intermédiaire. Les auteurs prouvent que si K' est de dimension finie à droite [resp. à gauche] sur L, K est galoisien sur K'. Ce

résultat avait été prouvé antérieurement par Nobusawa [Math. J. Okayama Univ. 7 (1957), 179–183; MR 20 #1697] mais avec l'hypothèse supplémentaire que K est localement fini à droite [resp. à gauche] sur L.

J. Dieudonné (Paris)

2632:

Moriya, Mikao. Zur Galoisschen Theorie der Schiefkörper. Math. J. Okayama Univ. 9 (1959/60), 49–62.

L'A. donne d'abord un nouveau critère pour qu'une sous-extension N d'une extension galoisienne finie (non commutative) K d'un corps non commutatif L soit une extension galoisienne de L: il faut et il suffit qu'il existe un groupe $\mathfrak{H}$ d'automorphismes de K dont N est l'ensemble des invariants, et un groupe $\mathfrak{G}$ d'automorphismes de K dont L est l'ensemble des invariants, tels que $\mathfrak{H}$ soit sous-groupe distingué de $\mathfrak{G}$. Il donne en suite (Satz 2) une formule pour le degré $[K:L]$ en fonction de $\mathfrak{G}$ et du commutant de L dans K, qui est essentiellement connue [voir Bourbaki, *Algèbre*, chap. 9, Hermann, Paris, 1959; MR 21 #6384; p. 122]. Enfin, il examine la structure de K lorsqu'on suppose que $\mathfrak{G}$ est un p-groupe, le centre de L ne contenant pas de racine p-ème de l'unité autre que 1 (p premier).

J. Dieudonné (Paris)

2633:

Tominaga, Hisao. A theorem on rings. Math. J. Okayama Univ. 9 (1959/60), 9–12.

The author proves the equivalence of the following three conditions for associative rings R. (1) For a fixed integer n and for every $x, y \in R$, the element $(xy-yx)^n-(xy-yx)$ belongs to the center Z of R. (2) For all integers m and for every $x, y \in R$, there exists an integer $n=n(x, y)$ such that $(xy-yx)^{nm}-(xy-yx) \in Z$. (3) All commutators $xy-yx$ of R are central and nilpotent. Conditions (2), (3) generalise a result of I. N. Herstein [Canad. J. Math. 9 (1957), 583–586; MR 19, 1035].

S. A. Amitsur (Jerusalem)

2634:

Baxter, Willard E. Concerning strong Lie ideals. Proc. Amer. Math. Soc. 11 (1960), 393–395.

Let A be a simple ring of characteristic $\neq 2$ or 3, either with its center $Z=(0)$ or of dimension greater than 16 over its center, and with an involution defined on it. Let S and K be the sets of symmetric and skew elements respectively. If U is an additive subgroup of K, it is called a Lie ideal of K if $[u, k]=uk-ku$ is in U for all u in U and k in K. Denote by $[K, K]$ the Lie ideal of K which is the additive subgroup generated by the elements of the form $[k, m]$ with k, m in K. A Lie ideal U of K is called a strong Lie ideal, if u in U implies u^3 is in U. The main result of this paper is: Let A be as described, and U a strong Lie ideal of K. Then either $U \subseteq Z$ or $U=K$. A subsidiary result of interest is: Let A be a ring of the type considered (except that characteristic 3 is now permitted). Then the finite dimensionality of either K, S, $[K, K]$ or $[K, S]$ implies the finite dimensionality of A.

K. G. Wolfson (New Brunswick, N.J.)

2635:

Amitsur, S. A. Finite dimensional central division algebras. Proc. Amer. Math. Soc. 11 (1960), 28–31.

Let D be a division ring, considered as an algebra over its center C. Then the polynomial ring $D[x]$ in a single commutative indeterminate x has zero Jacobson radical, so that $D[x]$ is a subdirect sum of primitive rings P_α (α running over some index set). The author's principal result is that D is finite over C if and only if these P_α are matrix rings of bounded order: more precisely, $(D:C) \leq n^2 < \infty$ if and only if every primitive homomorphic image of $D[x]$ is a complete matrix ring (over some division ring) of order $\leq n$. Necessity follows easily from the observation that every primitive image of $D[x]$ (except possibly $D[x]$ itself) is of the form $D \otimes_C E$ for some simple algebraic extension E of C; the crucial step in establishing the converse is the proof that D is algebraic over C, of degree $\leq n$.

Using his own theory of J-pivotal monomials [Proc. Amer. Math. Soc. 9 (1958), 635–642; MR 20 #3188; theorem 4], the author also proves the following: Let R be any principal right ideal ring without zero divisors; then every primitive homomorphic image of R is isomorphic with a complete matrix ring, over a division ring, of order $\leq n$, if and only if $1-ba^n$ and b are relatively left prime for all $a, b \in R$. {This is a corrected version of the author's theorem 3, whose statement contains two errors.} On taking $R=D[x]$, this gives an alternative criterion for $(D:C) \leq n^2$.

M. P. Drazin (Baltimore, Md.)

2636:

Faith, Carl. Algebraic division ring extensions. Proc. Amer. Math. Soc. 11 (1960), 43–53.

Given any ring A and any proper subring B, the author calls A radical over B if to each $a \in A$ corresponds a positive integer $n(a)$ such that $a^{n(a)} \in B$. This relationship has been considered by Kaplansky [Canad. J. Math. 3 (1951), 290–292; MR 13, 101], Herstein [ibid. 5 (1953), 238–241; MR 14, 719] and the reviewer [Rend. Circ. Mat. Palermo (2) 6 (1957), 51–64; MR 20 #3189] in the special case when B is central in A, and the author now investigates it in general, his principal result being that (*) if A is simple with minimal one-sided ideals, then A must be a commutative field. {Here the hypothesis of simplicity cannot be omitted or even weakened to that of primitivity; but it remains open how far the existence of one-sided ideals is essential.} As corollaries, the author deduces two results extending earlier work of Hua, Jacobson, Kaplansky and Wedderburn on division rings.

The author also considers extensions A/B of more general types. Given a field Φ, let S be a set of polynomials $f(x)=\sum_{i\geq 1} \alpha_i x^i \in \Phi[x]$ such that either $\alpha_1 \neq 0$ for all $f \in S$ or $\sum_i \alpha_i \neq 0$ for all $f \in S$. Call an algebra A over Φ S-algebraic over a given subalgebra B if to each $a \in A$ there corresponds $f_a(x) \in S$ such that $f_a(a) \in B$; and call S a $\blacktriangle$-set [$\bigstar$-set] if every division algebra [simple algebra with minimal condition] S-algebraic over a proper subalgebra is commutative. Then the author proves a division ring result of Nakayama type, and that every $\blacktriangle$-set is a $\bigstar$-set, and also the following: let S be a $\blacktriangle$-set and A any simple algebra (over Φ) S-algebraic over some proper subalgebra B. Then (i) (generalizing (*)) if A has a minimal one-sided ideal, A must be a commutative field; (ii) if $\Phi \neq GF(2)$ and every non-nil left ideal of A contains a non-zero idempotent, A must be either a commutative field or nil.

This review by no means exhausts the results and

concepts of this paper, to which the reader is accordingly referred.

{Reviewer's comment. The proof of Lemma 5 is incomplete, but, in reply to a letter from the reviewer, the author has bridged the gap as follows: The argument as given shows that $ab=ba$ for all $a \in A-B$, $b \in B$, whence it follows easily that B is commutative. But this implies that every B_β, and hence also A itself, is commutative; for either $B_\beta \leq B$ or else B_β is S-algebraic over the proper subalgebra $B \cap B_\beta$.} *M. P. Drazin* (Baltimore, Md.)

2637:

Utumi, Yuzo. A remark on quasi-Frobenius rings. Proc. Japan Acad. **36** (1960), 15–17.

Let A be a ring with unity satisfying the minimum conditions for left and right ideals. An A-module M satisfies condition (P) if whenever the two submodules N_1 and N_2 are isomorphic then so are the modules M/N_1 and M/N_2. The main result of the paper is as follows. Let A be a ring and B be a left A-module such that: (1) A is a direct summand of B; (2) for every indecomposable summand Ae of A, B contains a direct summand which is the direct sum of two isomorphic copies of Ae; and (3) B satisfies (P). Then A is quasi-Frobenius. *R. E. Johnson* (Rochester, N.Y.)

2638:

Dionísio, J. J. The concept of a semi-ring and its applications. Ciência. Lisboa No. 15/16 (1958/59), 43–51. (Portuguese)

Expository.

NON-ASSOCIATIVE RINGS AND ALGEBRAS

See also 2634.

2639:

Dorofeev, G. V. An instance of a solvable, though non-nilpotent, alternative ring. Uspehi Mat. Nauk **15** (1960), no. 3 (93), 147–150. (Russian)

A ring as described in the title is constructed. *R. A. Good* (College Park, Md.)

HOMOLOGICAL ALGEBRA

See 2989.

GROUPS AND GENERALIZATIONS

See also 2540, 2564.

2640:

Dlab, V. On a problem of Mazur and Ulam about irreducible generating systems in groups. Colloq. Math. **7** (1959/60), 171–176.

The author continues his study of generating systems [see Czechoslovak Math. J. **8** (83) (1958), 54–61; MR **20** #1707]. Examples of noncommutative groups G are given which possess property P: The group G has an irreducible (i.e., irredundant) generating system, but G has a subgroup H such that every generating system of H is redundant. Examples are constructed in which (a) H is a direct factor, (b) H is normal but not a direct factor, (c) H is not normal. A special case (given by the author) conveys the idea of his construction. Let H be the additive group of rational numbers; let A have an infinite irreducible generating system. Then $G=H\times A$ has property P. *J. L. Brenner* (Palo Alto, Calif.)

2641:

Baumslag, Gilbert. On a problem of Lyndon. J. London Math. Soc. **35** (1960), 30–32.

If $a^nb^n=c^n$, where $n>1$ and a, b, c are elements of a free group, then $ab=ba$. This extension of a result of the reviewer (for $n=2$) has been obtained independently by Schenkman, Schützenberger, and Stallings [see MR **21** #3476]. The problem is reduced to showing $a^pb^p=c^p$ impossible, for p a prime, in the free group on generators a_1 and b_1, where $a\equiv a_1$, $b\equiv b_1$, and hence $c\equiv a_1b_1$, modulo the commutator subgroup. This is done by showing that the p-Sylow group of the symmetric group of degree p^2 is generated by elements a_1' and b_1' such that the corresponding correspondences imply a' and b' of order p and c' of order p^2. *R. C. Lyndon* (London)

2642:

Fox, Ralph H. Free differential calculus. V. The Alexander matrices re-examined. Ann. of Math. (2) **71** (1960), 408–422.

[For part IV see R. H. Fox and K. T. Chen, same Ann. (2) **68** (1958), 81–95; MR **21** #1330.] The Alexander polynomial of a group G, finitely presented with more generators than relations, was defined, in the second of this series of papers [Fox, same Ann. (2) **59** (1954), 196–210; MR **15**, 931], only when the commutator quotient group H was torsion-free. This already gave difficulty in connection with the classification of lens spaces. Developing a remark of Blanchfield, the Alexander polynomial is here replaced by the more general concept of Alexander derivative.

Let $G=F/R$, F free on generators $x_1, \cdots, x_n$, f in JF, $r_1, \cdots, r_{n-1}$ in R, and let $E_0(H)$ be the order ideal of H. The determinant, evaluated in $JH/E_0(H)$, of

$$\partial(f, r_1, \cdots, r_{n-1})/\partial(x_1, \cdots, x_n),$$

depends only on the image in JH of f, and defines an 'elementary derivative' from JH into $JH/E_0(H)$. If the deficiency $d(G)$ (=maximum excess of generators over relations for a finite presentation) is positive, then $E_0(H)=0$ and the module of elementary derivatives is generated by a single 'Alexander derivative' ∇: JH into JH. If $d(G)>1$, $\nabla=0$. If $d(G)=1$ and H is torsion-free, ∇ is essentially the Alexander polynomial.

Presentations, deficiencies, Jacobians, and order ideals are defined for homomorphisms θ: G_1 into G_2, and a relation established of the form $(\nabla_0 v)^\rho\cdot\sigma=\nabla_1(v^\rho)$, for $E_0(\theta)=(\sigma)$. These ideas lead to the following theorem. Let M be a compact 3-manifold, with non-vacuous boundary containing no 2-sphere or projective plane, and K a simple closed polygon in the interior of M, representing k in $H(M)$. If θ is the inclusion map from $H(M-K)$ into $H(M)$, one has $(\nabla_{M-K}v)^\rho=(k-1)\nabla_M(v^\rho)$. This has as consequences a result on links of Torres [same Ann. (2) **57**

(1953), 57–89; MR 14, 574], and an extension to links of a result of Seifert [Quart. J. Math. Oxford Ser. (2) 1 (1950), 23–32; MR 11, 735] concerning knots contained within knotted tori. *R. C. Lyndon* (London)

2643:

Szép, J. Über eine allgemeine Erweiterung von Gruppen. II. Publ. Math. Debrecen 6 (1959), 254–261.

Let Γ be a set with a binary operation $\alpha\circ\beta$. Then Γ is called a near group if (i) Γ contains an element ε such that $\varepsilon\circ\alpha=\alpha=\delta\circ\varepsilon$ for all $\alpha\in\Gamma$, (ii) for any $\beta,\gamma\in\Gamma$ the equation $x\circ\gamma=\beta$ has a unique solution in Γ. Now let G be any group with a subgroup A and a system $\Gamma(A)$ of coset representatives (such that A is represented by the unit-element of G): $G=\bigcup\{A\alpha|\alpha\in\Gamma(A)\}$. If a multiplication is defined in $\Gamma(A)$ by the rule $\alpha\circ\beta=\gamma$, where $\alpha\beta=a\gamma$ $(a\in A)$, then $\Gamma(A)$ is a near group. The decomposition $G=(A,\Gamma(A))$ is an example of a general extension in the author's sense [cf. Szép, same Publ. 6 (1959), 60–71; MR 21 #3480]; it is called proper if A and $\Gamma(A)$ each have more than one element. The extension problem for such extensions is: given a group B and a near group F, to find all groups G with a decomposition $(A,\Gamma(A))$ such that $A\cong B$ and $\Gamma(A)\cong F$. It is solved by specializing the solution of the general extension problem [loc. cit.]. As an application the author proves the following generalization of known results: Let $G=AB$ be a factorization of a finite group such that $A\cap B$ is normal in B and either A is nilpotent and $B/A\cap B$ is abelian or vice versa; then G is soluble. *P. M. Cohn* (Manchester)

2644:

Fuchs, L. ★Abelian groups. International Series of Monographs on Pure and Applied Mathematics. Pergamon Press, New York - Oxford - London - Paris, 1960. 367 pp. $10.50.

A republication of the edition published in Hungary in 1958 and reviewed in MR 21 #5672.

2645:

Farahat, H. K. The symmetric group as metric space. J. London Math. Soc. 35 (1960), 215–220.

The problem which has aroused the author's interest is that of finding the order of the group of symmetries G of the polyhedron P in the space of all $n\times n$ matrices, having the permutation matrices as vertices. To solve this problem he generalizes it and defines the restricted symmetric group S on a set Z to be that normal subgroup S of the group of all permutations of the elements of Z composed of those that permute a finite number of such elements. He defines a metric in S and symmetry operations which preserve distances. By taking Z to be the set of n columns of the $n\times n$ identity matrix he shows that every symmetry of P corresponds to a symmetry of S and conversely, and concludes that the order of the group G of symmetries of P is $2(n!)^2$ except when $n=1, 2$. *G. de B. Robinson* (Toronto)

2646:

★Proceedings of Symposia in Pure Mathematics. Vol. I: Finite groups. American Mathematical Society, Providence, R.I., 1959. vii+110 pp. $3.90.

Eleven articles, to be reviewed individually.

2647:

Gorenstein, Daniel; Herstein, I. N. On the structure of certain factorizable groups. II. Proc. Amer. Math. Soc. 11 (1960), 214–219.

[For part I see same Proc. 10 (1959), 940–945; MR 22 #71.] The theorem proved is: Let the finite group G be a product AB, where A and B are cyclic. If $N(A)$ denotes the normaliser of A, let $N^*(A)$ be the upper bound of the groups $N(A)$, $N^2(A)$, $\cdots$. Then there is a unique cyclic normal subgroup T of G such that $G=N^*(A)T$ and $N^*(A)\cap T=1$; and if B^* is the subgroup of B such that $N^*(A)=AB^*$, T and B^* commute elementwise. *Graham Higman* (Chicago, Ill.)

2648:

Ignat'eva, R. P. Theorems on the existence and inclusion of subgroups in a finite group. Dokl. Akad. Nauk SSSR 134 (1960), 33–35 (Russian); translated as Soviet Math. Dokl. 1, 1011–1013.

A further generalisation of S. A. Čunihin's theorem 14 [Mat. Sb. (N.S.) 41 (83) (1957), 37–48; MR 19, 13] which is in itself quite general, containing as it does Sylow's theorem, P. Hall's theorem on the existence of subgroups in soluble groups and Schur's theorem on splitting extensions as special cases. *K. A. Hirsch* (St. Louis, Mo.)

2649:

Čunihin, S. A. Π-factorizations of finite groups with permutable factors. Mat. Sb. (N.S.) 50 (92) (1960), 383–388. (Russian)

This paper treats in detail results which were briefly announced earlier by the author [Dokl. Akad. Nauk SSSR 119 (1958), 888–889; MR 20 #3210]. *R. A. Good* (College Park, Md.)

2650:

Kazačkov, B. V. The Schur-Zassenhaus theorem for countable locally finite groups. Mat. Sb. (N.S.) 50 (92) (1960), 499–506. (Russian)

Let Π be a nonempty set of primes and $\mathfrak{G}$ a countable locally finite group. The Schur-Zassenhaus results are extended to $\mathfrak{G}$ in the following form. Suppose $\mathfrak{G}$ has an invariant Sylow Π-subgroup $\mathfrak{N}$. If $\mathfrak{N}$ is locally normal and locally solvable, then it may be complemented in $\mathfrak{G}$, and all its complements are locally conjugate to one another. If $\mathfrak{G}/\mathfrak{N}$ is locally normal and locally solvable, then $\mathfrak{N}$ may be complemented in $\mathfrak{G}$, and all its complements are locally normal, locally solvable, and locally conjugate to one another. *R. A. Good* (College Park, Md.)

2651:

Gorčakov, Yu. M. Primarily factorizable groups. Dokl. Akad. Nauk SSSR 134 (1960), 23–24 (Russian); translated as Soviet Math. Dokl. 1, 1001–1002.

A group is called completely factorizable if every subgroup of it is complemented. The structure of such groups is known [see N. V. Baeva-Černikova, same Dokl. 92 (1953), 877–880; Mat. Sb. (N.S.) 39 (81) (1956), 273–292; MR 15, 503; 18, 639]. In the present note the author considers primarily factorizable groups, i.e., groups in which every p-subgroup is complemented. He first shows that this class is, in fact, wider than that of the completely factorizable groups. His complicated example is a

suitable subgroup of the cartesian product of suitable dihedral groups. He then announces two theorems on the structure of periodic primarily factorizable groups. (1) A periodic subgroup of the cartesian product of completely primitive groups is primarily factorizable if and only if each of its p-projections is completely factorizable. (2) Every periodic primarily factorizable group is isomorphic to a subgroup of the cartesian product of completely primitive groups, generated by a primary complex of elements; and conversely, every periodic subgroup of such a cartesian product generated by a primary complex is primarily factorizable. Here a group is called completely primitive if it is a completely factorizable subgroup of the holomorph of a cyclic group of order p whose order is divisible by p. And the term p-projection is defined as follows: Take the cartesian product of completely primitive groups and in it the cartesian product of those factors that, for a fixed p, are p-primitive. For any complex of elements of the bigger product take their components in the smaller product. They form the p-projection of the complex.

K. A. Hirsch (St. Louis, Mo.)

2652:

Polovickiĭ, Ya. D. Layerwise extremal groups. Dokl. Akad. Nauk SSSR **134** (1960), 533–535 (Russian); translated as Soviet Math. Dokl. **1**, 1112–1113.

This note is based on the work of S. N. Černikov on groups with finite layers [Mat. Sb. (N.S.) **22** (**64**) (1948), 101–133; same Dokl. **70** (1950), 965–968; Mat. Sb. (N.S.) **45** (87) (1958), 415–416; MR **9**, 566; **11**, 496; **20** #5240]. The results are announced without proofs. Definitions: A group is layerwise finite if the number of elements of any given order is finite. Such a group is thin if all its Sylow subgroups are finite. A group is extremal if it is finite or a finite extension of a direct product of a finite number of quasicyclic groups. It is layerwise extremal if every set of elements of one and the same order generates an extremal group. A direct product of groups is primary thin if for every prime p there is only a finite number of direct factors containing p-elements. Let Π be a set of prime numbers. A group G is said to satisfy the Π-minimal condition for subgroups if every descending chain of subgroups $G_1 > G_2 > \cdots$ breaks off after a finite number of steps in which for every k the difference set $G_k \setminus G_{k+1}$ contains an element g_k such that $g_k^{n_k} \in G_{k+1}$ for a natural number n_k whose prime divisors all belong to Π. Such groups are obviously periodic, and the definition simplifies to the postulate that every $G_k \setminus G_{k+1}$ contains at least one Π-element. If a group satisfies the Π-minimal condition for every set Π consisting of a single prime number, it is said to satisfy the primary minimal condition. Theorem 1: Every locally finite group with Π-minimal condition is an extension of a group with Π-minimal condition and without proper subgroups of finite index by a group in which all Π-elements are contained in a finite normal subgroup. Theorem 2: A locally finite group G in which every subgroup without proper subgroups of finite index is abelian satisfies the Π-minimal condition if and only if the subgroup generated by all the Sylow Π-subgroups of G is extremal. Theorem 3: A locally finite group G is layerwise extremal if and only if every countable subgroup has a normal system with finite factors and satisfies the primary minimal condition. Theorem 4: A group can be embedded in a primary thin direct product of extremal groups if and only if it is layerwise extremal. This last theorem is an analog to S. N. Černikov's theorem [loc. cit.] on the embeddability of a thin layerwise finite group in a primary thin direct product of finite groups.

K. A. Hirsch (St. Louis, Mo.)

2653:

Suprunenko, D. A. Real linear locally nilpotent groups. Mat. Sb. (N.S.) **50** (**92**) (1960), 59–66. (Russian)

In an earlier paper [Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 273–274; MR **17**, 456] the author has determined all maximal irreducible locally nilpotent subgroups of the linear group GL(n, F), where F is an algebraically closed field. They form a single class of conjugates. In the present paper the author deals with the same question for the field R of real numbers. The main results are as follows: (1) When $n > 1$ is odd, then GL(n, R) contains no irreducible locally nilpotent subgroups. (2) When n is even, but not a power of 2, then the maximal irreducible locally nilpotent subgroups of GL(n, R) form again a single class of conjugates. (3) When n is a power of 2, then there are two conjugacy classes of maximal irreducible locally nilpotent subgroups. (4) For arbitrary $n > 2$ every irreducible locally nilpotent subgroup of GL(n, R) is imprimitive.

K. A. Hirsch (St. Louis, Mo.)

2654:

Osima, Masaru. On some properties of group characters. Proc. Japan Acad. **36** (1960), 18–21.

Following recent work by Brauer [Math. Z. **72** (1959/60), 25–46; MR **21** #7258], the author defines the section $S(P)$ of a p-element P of a finite group G as the set of all elements Q of G which can be written in the form $Q = P'R = RP'$, where P' is conjugate to P and R is p-regular. He proves in theorem 1 that if χ_i and χ_j belong to different blocks B and B' then $\sum \chi_i(Q)\bar{\chi}_j(Q) = 0$, summed over all $Q \in S(P)$. This result supplements Brauer's theorem that if Q and T belong to different sections of G then $\sum \chi_i(Q)\bar{\chi}_i(T) = 0$, summed over all $\chi_i \in B$. In particular, $\sum \chi_i(R)\bar{\chi}_j(R) = 0$, summed over all R, and this is generalized in theorem 3.

G. de B. Robinson (Toronto)

2655:

Fong, Paul. Some properties of characters of finite solvable groups. Bull. Amer. Math. Soc. **66** (1960), 116–117.

The author sketches the proof of a special case of a conjecture by Brauer that the defect group of order p^d of a given p-block B of a finite group G of order $g = p^a g_0$, $(p, g_0) = 1$, is Abelian if and only if the degree of every irreducible representation in B is divisible by exactly p^{a-d}. The case considered is when $a = d$.

G. de B. Robinson (Toronto)

2656:

Ševrin, L. N. On subsemigroups of free semigroups. Dokl. Akad. Nauk SSSR **133** (1960), 537–539 (Russian); translated as Soviet Math. Dokl. **1**, 892–894.

Let S be the free semigroup on a set of symbols $A \cup A^{-1} \cup B$, where the elements of A^{-1} are formal inverses of those of A. When B is empty, we have of course a free group, and when A is empty, we talk of a "pure" free semigroup. Since a subgroup of a free semigroup need not be free, the question arises under what additional restriction this is the case. In this note the author gives a

set of necessary and sufficient conditions. In the case of a pure free semigroup they can be stated very simply: A subsemigroup H of a pure free semigroup S is itself free if and only if for all $s \in S$ and $h \in H$ the relations $sh \in H$ and $hs \in H$ imply that $s \in H$. In the case of a mixed free semigroup they look even simpler: A subsemigroup H of a free semigroup S is itself free if and only if H is closed. But this simplicity is deceptive; for the definition of what a closed subsemigroup is, requires several other concepts and quite a bit of space. *K. A. Hirsch* (St. Louis, Mo.)

2657:

Chaudhuri, Niranjan Prasad. Sur les propriétés des demi-groupes inversifs vérifiant la règle de simplification d'un côté. C. R. Acad. Sci. Paris **250** (1960), 1421–1423.

Let S be a left cancellative, regular (inversif) semigroup. Let $H \subseteq S$. Then H is contained in a single equivalence class of the Dubreil principal right equivalence $\mathcal{R}_H$ if and only if H is right perfect. If H is unitary, then H is an equivalence class of $\mathcal{R}_H$. The author also shows that S is completely simple. {Reviewer's comment: In fact it is well known [cf. A. H. Clifford, Ann. of Math. (2) **34** (1933), 865–871; or H. B. Mann, Bull. Amer. Math. Soc. **50** (1944), 879–881; MR **6**, 147] that S is isomorphic to a direct product $G \times E$, where G is a group and E is a right zero semigroup, i.e., $ef=f$ for e, f in E. From this the unitary subsets and the right perfect subsets of S can be precisely determined.} *G. B. Preston* (Shrivenham)

2658a:

Livšic, A. H. Direct decompositions with indecomposable components in algebraic categories. Mat. Sb. (N.S.) **51 (93)** (1960), 427–458. (Russian)

2658b:

Livšic, A. H. Direct decompositions of idempotents in semigroups. Dokl. Akad. Nauk SSSR **134** (1960), 271–274 (Russian); translated as Soviet Math. Dokl. **1**, 1060–1063.

The first paper pursues the theory initiated by Kuroš [Trudy Moskov. Mat. Obšč. **8** (1959), 391–412; **9** (1960), 562; MR **21** #3365]; the second paper commences establishing the theory on one operation (composition) instead of two.

The Wedderburn-Remak-Schmidt Theorem is established for an object admitting a finite direct decomposition into indecomposable factors, and satisfying a condition (*) to the effect that an indecomposable direct factor must be subordinated to one or the other factor of any direct decomposition into two factors. It is extended to infinite direct decompositions under either of two additional conditions, with a weakened conclusion in one case; these results generalize the theorem of Kiokemeister [Bull. Amer. Math. Soc. **53** (1947), 957–958; MR **9**, 224] and, according to the author, a theorem of O. N. Golovin. The author concludes by deriving from his results the corresponding results on groups, including R. Baer's theorem [Trans. Amer. Math. Soc. **64** (1948), 519–551; MR **10**, 425] on groups satisfying a "splitting hypothesis" like (*): any two indecomposable direct decompositions are centrally isomorphic; and deriving also the generalized Krull-Schmidt theorem of Atiyah [Bull. Soc. Math. France **84** (1956), 307–317; MR **19**, 172].

The second paper concerns semigroups, or equivalently, categories without an extra operation of addition. The extra operation is then defined, restricted to direct summation of a family of idempotents. Nothing is said about whether (or when) the operation will satisfy Kuroš's axioms, nor are any details given concerning applications. The theorems stated appear to generalize all the main results of the first paper. The definition of "normal" is incorrect; see the first paper for the correct definition.
J. Isbell (Seattle, Wash.)

2659:

Kertész, A.; Sade, A. On nuclei of groupoids. Publ. Math. Debrecen **6** (1959), 214–233.

The authors first generalize the concepts of nucleus and center of a groupoid [Bruck, *A survey of binary systems*, Springer, Berlin, 1958; MR **20** #76]. Let G be a set and ψ_1, ψ_2: $G^k \to G$ two functions of k variables. For each i $(1 \leq i \leq k)$, the set $X_i \subset G$ consisting of those $x \in G$ such that

$$\psi_1(a_1, \cdots, a_{i-1}, x, a_{i+1}, \cdots, a_k) = \psi_2(a_1, \cdots, a_{i-1}, x, a_{i+1}, \cdots, a_k)$$

for all $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_k \in G$ is the ith nucleus defined by ψ_1, ψ_2. For example, if $(G, \circ)$ is a groupoid defined on G, $\psi_1(a_1, a_2, a_3) = (a_1 \circ a_2) \circ a_3$, $\psi_2(a_1, a_2, a_3) = a_1 \circ (a_2 \circ a_3)$, then $X_1 \cap X_2 \cap X_3$ is the usual nucleus of G. The authors examine the X_i's defined by other choices of ψ_1 and ψ_2.

In a similar spirit they introduce 'endomorphizer' nuclei associated with a function $\varphi: G \to G$. For example, the left-endomorphizer nucleus of φ is the set of $x \in G$ such that $\varphi(x)\varphi(a) = \varphi(x \circ a)$ for all $a \in G$. They examine these nuclei for special φ's and G's.
S. Stein (Davis, Calif.)

2660:

Osborn, J. Marshall. Loops with the weak inverse property. Pacific J. Math. **10** (1960), 295–304.

A loop is said to have the weak inverse property if, whenever three of its elements, x, y, z, satisfy $xy \cdot z = 1$, they also satisfy $x \cdot yz = 1$. Inverse property loops and crossed inverse property loops [for the definitions see, e.g., Bruck, *A survey of binary systems*, Springer, Berlin, 1958; MR **20** #76] have the weak inverse property. The author proves that in a weak inverse property loop the left, right and middle nuclei coincide, thus generalizing the theorems stating the same fact for inverse property loops [Bruck, loc. cit.] and crossed inverse property loops [Artzy, Trans. Amer. Math. Soc. **91** (1959), 480–492; MR **21** #5688]. Another known fact concerning inverse property [crossed inverse property] loops G is that, if every loop isotopic to G has the inverse [crossed inverse] property, then G is Moufang [Bruck, Artzy, loc. cit.]. The following theorems of the paper constitute generalizations of this result: (1) If G is an inverse property, cross inverse property or commutative loop such that every isotope has the weak inverse property, then G is Moufang. (2) If G is a loop all of whose isotopes have the weak inverse property and if N is its nucleus, then N is normal and G/N is Moufang.

Furthermore, the construction of a class of loops on one generator is described, having the property that each

of their isotopes has the weak inverse property. Every isotope of a loop of this class is isomorphic to the loop itself. These loops do not have the Moufang property, while the only previously known examples of loops isomorphic to all their isotopes were Moufang loops.
R. Artzy (Chapel Hill, N.C.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 2972.

2661:

Ghika, Al. Groupes à racines topologiques localement paraconvexes. I. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 27–42.

The author's aim is to extend certain basic results on locally convex topological vector spaces to the so-called locally paraconvex topological groups, which are in general non-abelian but have the defining property of the divisible abelian groups. However the part under review contains only algebraic preliminaries. It deals exclusively with definitions and elementary properties of convex sets, absorbant sets, etc., in a group. *Ky Fan* (Detroit, Mich.)

2662:

Semadeni, Z. Sur les groupes métriques complets. Colloq. Math. **7** (1959), 35–39.

Studying metrizable topological groups which are complete or topologically complete, the author deduces several results from the theorem of Mazur and Sternbach to the effect that a G_δ subgroup of a metric group must be closed. He observes, in particular, that a group which is metrizable and topologically complete must be absolutely closed. Though most of his results are at least partly known, the discussion is interesting. [See also a recent paper by Weston, Arch. Math. **10** (1959), 40–41; MR **21** #3832.] *Victor Klee* (Seattle, Wash.)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

2663:

Gluskin, L. M. The semi-group of homeomorphic mappings of an interval. Mat. Sb. (N.S.) **49 (91)** (1959), 13–28. (Russian)

Let E be the closed unit interval and S the set of all strictly monotonic continuous functions from E into E; under usual composition of functions S is a semigroup. The paper treats ideals in various subsemigroups of S, their automorphisms and topology.

In order to cite some of the results we need the following definitions: $F=\{0,1\}$; $K=\{x\colon\ x\in S,\ F\cap x(E)=\emptyset\}$; $G=\{x\colon x\in S,\ x(F)=F\}$; $J=S-G$; for each $[\alpha,\beta]\subset E$, $R(\alpha,\beta)=\{x:x\in S, x(E)\subset[\alpha,\beta]\}$; $D=\{x:x\in S,\text{ increasing}\}$; $B_0=\{x\colon x\in D,\ x(0)=0,\ x(1)<1\}$.

Among the algebraic results are: J, K, S are the only 2-sided principal ideals in S, and $R(\alpha,\beta)$ the only right-principal ideals; each automorphism of B_0, D or S is of the form $x\to fxf^{-1}$ for a suitable $f\in S$.

A subset B of a topological semigroup T is a basis for T if T is the topological closure of the algebraic closure of B. It is proved that J has no finite basis but that S does (this uses a result of Schreier and Ulam, namely, G has a finite basis).

The author remarks that some of the results generalize to the n-cell. It is also true that some hold for ordered sets other than E. *S. Stein* (Davis, Calif.)

FUNCTIONS OF REAL VARIABLES

See also 2810, 2890, B3603.

2664:

Sengupta, H. M.; Ganguli, P. L. A class of steadily increasing continuous functions which are not absolutely continuous. Bull. Calcutta Math. Soc. **51** (1959), 53–56.

Analytic description of a process for constructing from any convergent infinite series of positive terms a monotone, continuous, non-absolutely-continuous function on the unit interval. *T. A. Botts* (Charlottesville, Va.)

2665:

Marcus, S. Conditions d'équivalence à une constante pour les fonctions intégrables Riemann et pour les fonctions jouissant de la propriété de Baire. Rev. Math. Pures Appl. **4** (1959), 283–285.

The theorem that if $f(x)$ is Riemann integrable on $R\equiv(-\infty,\infty)$ and $\int_{-\infty}^{+\infty}|f(x+h)-f(x)|dx=0$ for h on a set H dense in R, then $f(x)=c$ almost everywhere, is proved by using a sequence $\{d_n\}$ in H dense in H. The same general procedure gives the result that if $f(x)$ has the Baire property on R and for h on H dense in R, $f(x+h)=f(x)$ except for a set of the first category, then $f(x)=c$ except for a set of the first category. {In the proof of the first theorem the symbol $\in$ should be replaced by $\notin$ in l. 4 and l. 6 on p. 284.}
T. H. Hildebrandt (Providence, R.I.)

2666:

Kabakov, F. A. Limit properties of the second finite difference. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 115–127. (Russian)

2667:

Videnskiĭ, V. S. Generalizations of Markov's theorem on the evaluation of a polynomial derivative. Dokl. Akad. Nauk SSSR **125** (1959), 15–18. (Russian)

Soient $a_k\geq 0$, $k=1, 2, \cdots, m$, $G(x)=\prod_1^m(1+a_k^2x^2)$, $\alpha_k=\sqrt{(1+a_k^2)}$ $(k=1, 2, \cdots, n)$; $\alpha_{m+1}=\cdots=\alpha_n=0$, $n\geq m$; posons

$$M_n(x) = \operatorname{Re}\prod_1^n(\alpha_kx+i\sqrt{(1-x^2)}),$$

$$\sqrt{(1-x^2)}N_{n-1}(x) = \operatorname{Im}\prod_1^n(\alpha_kx+i\sqrt{(1-x^2)}),$$

où les M_n et N_{n-1} sont des polynômes de degré n, resp. $n-1$. Si le polynôme $P_n(x)$ de degré $n\geq m$ satisfait à l'inégalité

$$(1)\quad |P_n(x)| \leq \sqrt{G(x)} = \prod_1^m|a_kx+i\sqrt{(1-x^2)}|,\quad -1\leq x\leq 1,$$

alors on a

$$(2)\quad |P_n'(x)| \leq M_n'(1) = n\alpha_1\alpha_2\cdots\alpha_n+2\sum\alpha_1\alpha_2\cdots\alpha_{n-2},\quad -1\leq x\leq 1,$$

où $\sum$ désigne la fonction symétrique de α_ν. Le signe d'égalité dans (2) a lieu seulement dans le cas où $P_n(x)=\gamma M_n(x)$, $|\gamma|=1$ et aux points $x=\pm 1$. Ce théorème se réduit pour $\alpha_1=\alpha_2=\cdots=\alpha_n$ au théorème de A. A. Markoff [G. Pólya et G. Szegö, *Aufgaben und Lehrsätze aus der Analysis, Bd. II*, 2te Aufl., Springer-Verlag Berlin, 1954; MR **15**, 512; VI Abschn., No. 83, p. 91], le second membre dans (2) devient alors égale à $T_n'(1)=n^2$ avec $T_n(x)=\cos n \arccos x$, tandis que $G(x)\equiv 1$.

M. Tomić (Belgrade)

2668:

Arnol'd, V. I. On functions of three variables. Dokl. Akad. Nauk SSSR **114** (1957), 679–681. (Russian)

Brief exposition of proof that any continuous real function $f(x_1, x_2, x_3)$ defined on the unit 3-cube can be represented in the form

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} h_{ij}[\varphi_{ij}(x_1, x_2), x_3],$$

where h_{ij} and φ_{ij} are continuous real functions of two variables.

This result solves the famous "13th problem of Hilbert", in the sense of refuting the conjecture there stated. The corresponding result for functions of more than 3 variables was obtained by A. N. Kolmogorov [same Dokl. **108** (1956), 179–182; MR **18**, 197].

L. D. Kudryavcev (RŽMat **1958** #2837)

2669:

Kolmogorov, A. N. On the representation of continuous functions of many variables by superposition of continuous functions of one variable and addition. Dokl. Akad. Nauk SSSR **114** (1957), 953–956. (Russian)

It is proved that, for any integer $n\geqq 2$, there exist continuous monotone-increasing real functions $\psi^{pq}(x)$ ($p=1, 2, \cdots, n$; $q=1, 2, \cdots, 2n+1$) defined on the interval $[0, 1]$, such that every continuous function $f(x_1, \cdots, x_n)$ defined on the unit n-cube can be represented in the form

$$(1)\qquad f(x_1, \cdots, x_n) = \sum_{q=1}^{q=2n+1} \chi_q\Big(\sum_{p=1}^{n} \psi^{pq}(x_p)\Big),$$

where $\chi_q(y)$ are also real continuous functions.

This theorem does not imply all the results of the author's earlier work in this direction [same Dokl. **108** (1956), 179–182; MR **18**, 197], but does contain their principal import from the point of view of representing functions of finitely many variables by superpositions of functions of fewer variables, and of approximating them by superpositions of the given form of polynomials of one variable and the operation of composition. The method of proof of the new theorem departs from the elementary methods of earlier works on this question [op. cit.; and V. I. Arnol'd, see preceding review]; however, the constructions applied to the proof of the new theorem are arrived at by analysis of the constructions used in the cited works, discarding from them details not needed to get the final result.

Taking, for $n=3$,

$$\varphi_q(x_1, x_2) = \psi^{1q}(x_1)+\psi^{2q}(x_2),\quad h_q(y, x_3) = \chi(y+\psi^{3q}(x_3)),$$

it follows from (1) that

$$(2)\qquad f(x_1, x_2, x_3) = \sum_{q=1}^{7} h_q[\varphi_q(x_1, x_2, x_3)].$$

Accordingly for $n=3$ the result obtained is a strengthening of the corresponding result of V. I. Arnol'd [review above], in that the number of terms on the right-hand side of (2) is reduced from nine (as in Arnol'd's paper) to seven.

L. D. Kudryavcev (RŽMat **1959** #1339)

2670:

Saunders, Herbert. A further extension of the Routh-Hurwitz stability criteria for quintic and sextic equations. J. Aero/Space Sci. **27** (1960), 396–397.

This is an application to fifth and sixth degree equations of the well-known stability criteria of Routh and Hurwitz, and in particular of that of Bilharz-Frank.

E. Frank (Chicago, Ill.)

MEASURE AND INTEGRATION

See also 2908, 2955.

2671:

Takahashi, Shigeru. A remark on the Riemann-sum. Sci. Rep. Kanazawa Univ. **6** (1959), 57–59.

For any function $f\in L_p[0, 1]$ of period 1, let $F_n(t, f)=(1/n)\sum_1^n f(t+i/n)$. When f is continuous, $\lim_{n\to\infty} F_n(t, f)$ exists and is $\int_0^1 f$, for every t. Jessen [Ann. of Math. (2) **35** (1934), 248–251] showed that if, for example, $n_k=k!$, then $\lim_{k\to\infty} F_{n_k}(t, f)=\int_0^1 f$ for any f, and for almost all t. In the present paper, the author investigates the sharpness of such theorems; he proves that if $\{n_k\}$ is an increasing sequence of primes, then there exists a function f and a real sequence $\{h_k\}$ so that $\lim_{k\to\infty} F_{n_k}(t+h_k, f)$ fails to exist for almost all t. This result is related to a theorem in the opposite direction [see S. Yano, Tôhoku Math. J. (2) **2** (1950), 1–3; MR **12**, 811].

R. C. Buck (Madison, Wis.)

2672:

Goffman, C.; Zink, R. E. Concerning the measurable boundaries of a real function. Fund. Math. **48** (1959/60), 105–111.

Making use of the complete lattice property of the extended-real valued measurable functions on a totally σ-finite measure space, the authors generalize to arbitrary extended-real valued functions on such a space H. Blumberg's definitions [Acta Math. **65** (1935), 263–282] of upper and lower measurable boundaries of extended-real valued functions on the real line. The authors then obtain, in the generalized setting and with simplified proofs, Blumberg's application of these notions to the proof of a theorem of Saks and Sierpiński's [Fund. Math. **11** (1928), 105–112] on the approximation of arbitrary functions by measurable functions.

T. A. Botts (Charlottesville, Va.)

2673:

Goffman, C.; Waterman, D. On upper and lower limits in measure. Fund. Math. **48** (1959/60), 127–133.

For sequences $\{f_n\}$ of extended-real valued measurable functions on the real interval $[a, b]$, the notions of upper and lower limit in measure were originally defined and studied by D. E. Men'šov [Trudy Mat. Inst. Steklov **32** (1950); Amer. Math. Soc. Transl. No. 105 (1954); MR **12**, 254; **15**, 866]. The present paper introduces equivalent but simplified definitions of these notions. The new definitions permit substantially simpler proofs of the basic properties. In studying stochastic convergence of martingales, K. Krickeberg [Math. Z. **66** (1957), 470–486; MR **19**, 948] defined upper and lower limits in measure for directed sets of measurable functions. The authors' definitions are similar to Krickeberg's and are in fact equivalent to Krickeberg's as specialized to sequences.

T. A. Botts (Charlottesville, Va.)

2674:

Lahiri, B. K. A property of sets of positive measure. Bull. Calcutta Math. Soc. **51** (1959), 79–86.

By analogy with the universal sets of H. Kestelman [J. London Math. Soc. **22** (1947), 130–136; MR **9**, 274], the author terms a set E in Euclidean n-space E_n universally permeable if for every sequence $\{\alpha_k\}$ of positive numbers convergent to 1 there is a vector ξ in E_n such that for all sufficiently large k the vectors ξ/α_k all belong to E. Following the methods of Kestelman [loc. cit.] this analogue of a result of Kestelman's is proved: if I is an interval and $S \subset I$ is either of measure zero or of first category, then $I - S$ is universally permeable.

T. A. Botts (Charlottesville, Va.)

2675:

Rutovitz, D.; Pauc, C. Y. Theory of Ward for cell functions. I, II. Ann. Mat. Pura Appl. (4) **47** (1959), 1–57.

{Part I is by Rutovitz, part II by Pauc.} The objective in these two papers is to extend theorems of A. J. Ward [Fund. Math. **28** (1936), 265–279; also S. Saks, *Theory of the integral*, Stechert, New York, 1937; pp. 265–279] on derivatives of additive interval functions in m-dimensional Euclidean space R_m to the case when R_m is replaced by an abstract set R. Assumed is a class **M** of subsets of R which form a σ-algebra, and a σ-finite, σ-additive measure function μ on **M** which gives rise to an outer measure $\bar{\mu}$, measurable covers $\hat{X}$ of X, and a class **N*** of null sets N^*. Nonoverlap is equivalent to $\bar{\mu}(A \cap B) = 0$. Intervals are replaced by a class **I** of subsets of **M** with finite positive measures, called cells. A complex consists of a finite number of nonoverlapping cells, and a figure is the union of cells of a non-empty complex. Partitions **P** of R consist of a denumerable number of nonoverlapping cells I_n such that $\bigcup_n I_n = R$, any cell I_n of **P** meets only a finite number of other cells of **P**. **P**′ is a refinement of **P**, if every cell of **P** consists of a finite number of cells of **P**′. Partitions can be ordered by successive refinements. A class of cells each of which belongs to some partition **P** of R is called a net. A net is normed if there exists a positive valued function δ on cells such that $I_1 \subset I_2$ implies $\delta(I_1) \leq \delta(I_2)$. A net has the U-property if for almost all points x of any cell I, all cells I_x containing x with sufficiently small $\delta(I_x)$ lie in I. For any set E, the set E_I consists of the points x of E for which there exists a sequence of cells I_n such that $\lim_n \delta(I_n) = 0$. Derivates of functions $F(I)$ of cells are defined at points x of E_I by convergence of $F(I_n)/\mu(I_n)$ as $\delta(I_n) \to 0$, I_n containing x. The following theorem generalizes Ward's theorem on strong derivates: If **I** is a U-normed net, having the p-bordering property and the weak Vitali property, then the additive cell function $F(I)$ has a derivative almost everywhere on any subset E_0 of E_I where both extreme derivates are finite. The p-bordering property defies abbreviation. Essentially it provides for the existence of partitions **P** such that certain complexes K of cells in **P** are contained in larger complexes K^b of cells of **P** with $\mu(K^b) < p\mu(K)$, $p > 1$. The weak Vitali property allows for small overlap of the finite number of cells involved in the conclusion of the property, the strong property requires nonoverlapping. In order to obtain the corresponding theorem where the assumption that both extreme derivates be finite is replaced by the assumption that one of the extreme derivates be finite, derivation is restricted to graduated **A**-nets, a nonempty family of sets in **M** of finite, positive measure, which can be represented in the form $\bigcup_{r=1}^{\infty} \mathbf{P}_r$, $\mathbf{P}_r$ being successive refinements of a partition $\mathbf{P}_1$ of R, which are subject to the condition that for some $\alpha > 0$, the $\inf \mu(I_{r+1})/\mu(I_r) > \alpha$, for all I_r in $\mathbf{P}_r$, I_{r+1} in $\mathbf{P}_{r+1} \cdot I_r$ and all r. Further restrictions are imposed on the net **A**, so that $\bar{D}_A F = \bar{D}_I F$ for almost all points of E_A for which $\bar{D}_A F < \infty$ (and similarly for $\underline{D}_A F$).

The paper of Pauc gives a proof of the first theorem above through the medium of Radon-Nikodym (global) derivates, being an exposition of an earlier note [Pauc and Rutovitz, C. R. Acad. Sci. Paris **240** (1955), 1956–1958; MR **17**, 832]. **M** is assumed to be the Borel extension of **I**, $R_I = R$. Then the main theorem asserts that if F is a finitely additive cell function, with finite extreme derivates on E, there exists a μ-measurable point function f, defined on an **M**-set containing E (mod **N***), unique (mod **N***), such that any set S in E with $\bar{\mu}(S) > 0$ includes a subset L contained in a figure, such that $\bar{\mu}(L) > 0$, f is defined and bounded (mod N) on a Jordan cover $\hat{L}$ of L and such that $F(I) = \int_{\hat{L} \cdot I} f d\mu + \int_I F(J \| L)$. Jordan covers are based on enclosures by a finite number of cells; the function $F(J \| L) = 0$ if J intersects L and $= F(J)$ if J does not intersect L; and $\int_I F(J \| L)$ is a Burkill set integral. If E is included in a figure, A, and **P** stands for a generic cell partition of A, then the function f is the limit in measure, as $\delta(\mathbf{P}) \to 0$, of the step functions $f_P(x) = F(I)/\mu(I)$ if I is in **P** and contains x [stochastic derivative of K. Krickeberg, Math. Nachr. **18** (1958), 203–217; MR **21** #114]. If almost every point of any **M**-set has δ density $= 1$ almost everywhere, or the weak Vitali property holds, then the integrand f agrees with the (existing) derivative of F almost everywhere on E.

T. H. Hildebrandt (Providence, R.I.)

2676:

Richards, Ian. A note on the Daniell integral. Rend. Sem. Mat. Univ. Padova **29** (1959), 401–410.

The note considers changes in the Daniell theory of the general integral [Ann. of Math. (2) **19** (1917/18), 279–294] if the basic integration process operates on a vector space which need not be a vector lattice. A type of Fubini theorem is proved. There are points of contact with the general integral of M. H. Stone [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 336–342, 447–455, 483–490; MR **10**, 24, 107, 239].

T. H. Hildebrandt (Providence, R.I.)

2677:

Halmos, Paul R. ★Lectures on ergodic theory. Chelsea Publishing Co., New York, 1960. vii+101 pp. $2.95.

This is a reprint of the book originally published in Japan, which was reviewed as MR **20** #3958.

2678:

Gladysz, S. Ein ergodisches Paradoxon. Colloq. Math. 7 (1959/60), 245–249.

Let (S, Σ, m) (reviewer's notation) be a finite measure space, and let ρ be a metric in S which is measurable with respect to $m\times m$. Suppose that h is a metrically transitive measure-preserving transformation in S and that $\int \rho(s, h(s))mds<\infty$. Let E be any measurable set. Then for any $s\in S$, we observe that $h^n(s)\in E$ for some values of n, call them $n_1, n_2, \cdots$. The points $h^{n_{i-1}}(s)$, $h^{n_{i-1}+1}(s), \cdots$, $h^{n_i}(s)$ form a "path" of "length" equal to

$$r_i(s) = \rho(h^{n_{i-1}}(s), h^{n_{i-1}+1}(s))+\cdots+\rho(h^{n_i-1}(s), h^{n_i}(s)).$$

It is quickly seen, by using the pointwise ergodic theorem, that the limit $\rho_1(s)=\lim_{k\to\infty} k^{-1}\sum_{i=1}^k r_i(s)$ exists and is equal to $\int \rho(s, h(s))mds/m(E)$ for almost all choices of s. If we define $\rho_n(s)=\lim_{k\to\infty} k^{-1}\sum_{i=1}^k r_{ni}(s)$, then the value of $\rho_n(s)$ may depend on s, but we have theorem 1: the function $\rho_n(s)$, defined as above, cannot take essentially more than n values.

It is not hard to construct examples where n values are actually taken. The author, however, proves a much stronger converse (theorem 2): If $0<m(F)<\frac{1}{2}m(S)$, then there is a measurable set E with the property that the function $\rho_2(s)$ defined above takes 2 values, one on F and the other on F'. *A. Beck* (Madison, Wis.)

2679:

Baiada, E. Sulla convergenza in lunghezza delle medie integrali. Ann. Mat. Pura Appl. (4) **48** (1959), 223–228.

If C: $x=x(t)$, $y=y(t)$, $a\leq t\leq b$, denotes any continuous rectifiable curve of length $L(C)<+\infty$, then $x(t)$, $y(t)$ are BV functions in $[a, b]$. If I_h denotes the integral $I_h=\int_a^{b-h}(x_h'^2+y_h'^2)^{1/2}dt$, where x_h, y_h denote the mean integrals $h^{-1}\int_0^h x(t+u)du$, $h^{-1}\int_0^h y(t+u)dt$, $a\leq t\leq b-h$, then it is known that $\lim I_h=L(C)$ as $h\to 0+$ [see, e.g., T. Radó, *Length and area*, Amer. Math. Soc., New York, 1948; MR **9**, 505; p. 207]. The author discusses the analogous integral $J_h=\int_a^{b-h}(x'^2+y_h'^2)^{1/2}dt$, and proves that $\lim J_h=L(C)$ as $h\to 0$ provided $x(t)$ is AC in $[a, b]$. If this last condition is not satisfied, then the author shows, by an example, that $\lim J_h$ may exist and be larger than $L(C)$. *L. Cesari* (Ann Arbor, Mich.)

2680:

Picone, Mauro. "Lunghezza" per un insieme di elementi di un arbitrario spazio metrico e sua semicontinuità inferiore. Math. Z. **73** (1960), 366–373.

The author considers mappings f from a subset A of an arbitrary space Σ into a metric space U with metric d, a family $\{X\}$ of subsets of Σ, and a mapping $X(x)$ of A into $\{X\}$ such that $x\in X(x)$. {The first sentence beginning on p. 367 seems to indicate that no x belongs to more than one X, but this is inconsistent with the use made of the mapping.} A relation—x precedes x'—is defined to mean: there exists a finite sequence (x_k) in A such that $x_0=x$, $x_n=x'$, and $x_k\in X(x_{k-1})$ for $k=1, \cdots, n$. If x precedes x', the $\mathrm{arc}_A(x, x')$ is defined as the subset of A consisting of x, x', and all x'' such that x precedes x'' and x'' precedes x'. The family $\{X\}$ is said to be properly distributed on A if the relation "precedes" is antisymmetric. Now consider finite sequences $S=(x_k)$ such that, for given x and x', $x_0=x$, $x_n=x'$, x_{k-1} precedes x_k for $k=1, \cdots, n$. Set $v_f(x, x')=\sup\sum_k d[f(x_{k-1}), f(x_k)]$ for all such S, and $v_f(A)=\sup v_f(x, x')$ for all x and x' in A with x preceding x'. Let F denote the image of A by f, and $\mathrm{arc}_F(x, x')=\mathrm{arc}[f(x), f(x')]$ denote the image of $\mathrm{arc}_A(x, x')$. Set length $F=v_f(A)$, length $\mathrm{arc}[f(x), f(x')]=v_f(x, x')$. The theorem is that the length of arc is a lower semicontinuous function of F in terms of the usual distance in the space of functions f. *L. M. Graves* (Chicago, Ill.)

2681:

Goffman, Casper. A remark concerning the area of a nonparametric surface. Proc. Amer. Math. Soc. **11** (1960), 463–467.

Let P be the set of quasi-linear real functions $p(x, y)$ on the unit square. It is proved that if Δ is any metric on P whose topology is coarser than that of uniform convergence but finer than that of convergence in measure, then the elementary area E is a Δ-lower semicontinuous function on P. Moreover the lower semicontinuous Fréchet extension of E to the Δ-completion of P agrees for continuous functions $f(x, y)$ with the Lebesgue area of f. *W. H. Fleming* (Providence, R.I.)

2682:

Král, Josef. On the Lebesgue area of closed surfaces. Czechoslovak Math. J. **9 (84)** (1959), 470–471. (Russian. English summary)

Let S be a compact, finitely triangulable $(n-1)$-manifold in euclidean n-space E_n, and $L(S)$ its Lebesgue area. Let $\mathfrak{A}^*$ denote the class of Lebesgue measurable sets A such that the gradient (in distribution theory sense) of the characteristic function of A is a totally finite measure. For $n=3$ the following results are announced. Theorem I: If there exists a set $A\in\mathfrak{A}^*$ which contains one of the two components of E_3-S and does not meet the other, then $L(S)$ is finite. Theorem II: If both components of E_3-S belong to $\mathfrak{A}^*$, then S has 3-dimensional Lebesgue measure 0. {Theorem I was proved independently by the reviewer [Illinois J. Math. **4** (1960), 452–478; p. 472], for $n=3$, and for $n>3$ in case S has n-dimensional measure 0. The reviewer also showed that if S has n-dimensional measure 0 and $L(S)<\infty$, then $L(S)$ equals the total variation of the gradient measure. This answers a question posed by H. Federer [Proc. Amer. Math. Soc. **9** (1958), 447–451; MR **20** #1751; p. 451]. Theorem II was also found by Federer for $n=3$, and remains an open question for $n>3$.} *W. H. Fleming* (Providence, R.I.)

FUNCTIONS OF COMPLEX VARIABLES

See also 2787, 2912, 2917.

2683:

Dzyadyk, V. K. Geometrical definition of analytic functions. Uspehi Mat. Nauk **15** (1960), no. 1 (91), 191–194. (Russian)

The following is proved: Let $u(x, y)$, $v(x, y)$ be real functions of class C' in a domain G. A necessary and sufficient condition that $u+iv$ be analytic or conjugate analytic in G is that the three surfaces defined by $z=u(x, y)$, $z=v(x, y)$, $z=\sqrt{(u^2+v^2)}$ have the same area over every subdomain g of G.

J. B. Crabtree (Hoboken, N.J.)

2684:

Bilimovitch, Anton. Sur les lignes principales des fonctions non analytiques. C. R. Acad. Sci. Paris **250** (1960), 805–807.

Es werden Bemerkungen gemacht zu den vom Verf. so genannten "lignes principales des fonctions non-analytiques", womit er folgendes meint: Es sei f eine Abbildung der Klasse C^1 von der z-Ebene in die w-Ebene, mit positiver Funktionaldeterminante. Einem infinitesimalen Kreis mit dem Zentrum z entspricht eine infinitesimale Ellipse mit dem Zentrum $w=f(z)$. Ihre Hauptachsen definieren in der w-Ebene zwei orthogonale Kurvenscharen. Analog werden in der z-Ebene zwei solche Kurvenscharen definiert.

A. Pfluger (Zürich)

2685:

Ikoma, Kazuo. On a property of the boundary correspondence under quasiconformal mappings. Nagoya Math. J. **16** (1960), 185–188.

Es gibt eine quasikonforme Abbildung f der obern Halbebene $\operatorname{Im} z \geqq 0$ auf sich, und eine Lebesgue'sche Nullmenge E auf der reellen Achse, deren Bild $f(E)$ ein positives Mass hat. Dieses bekannte Resultat von Beurling und Ahlfors wird durch das folgende Beispiel des Verf. kontrastiert: Es gibt auf der reellen Achse eine Menge E vom Lebesgueschen Mass null, deren Bild $f(E)$ bei jeder quasikonformen Abbildung f der obern Halbebene auf sich wieder vom linearen Mass null ist. Diese Menge E ist ein Cantor'sches Diskontinuum $E(p_1, p_2, \cdots)$ von positiver logarithmischer Kapazität. *A. Pfluger* (Zürich)

2686:

Havin, V. P. On the norms of some operations in the space of polynomials. Vestnik Leningrad. Univ. **14** (1959), no. 19, 47–59. (Russian. English summary)

The author investigates relations between the maximum of a polynomial on a curve in the complex plane and certain weighted sums of its coefficients. Let L be a rectifiable closed Jordan curve in the plane, and P_n the space of all complex polynomials of degree no higher than n. We introduce into P_n the norm $\|p\|_L = \sup_{z\in L} |p(z)|$. Define U_n to be the linear operator on P_n which sends $p(z)=\sum_{k=0}^{n} c_k z^k$ into the $(n+1)$-termed sequence $c=(c_0, c_1, \cdots, c_n)$. We shall consider "norms" for U_n of the form

$$\|U_n\|_{d,l_p} = \sup_{\|p\|_L \leq 1} \|U_n p\|_{d,l_p},$$

where p is a fixed positive number, $d_0, d_1, \cdots, d_n, \cdots$ is a fixed infinite sequence of non-negative numbers, and, for each $(n+1)$-termed sequence $c=(c_0, c_1, \cdots, c_n)$ of complex numbers,

$$\|c\|_{d,l_p} = (\sum_{k=0}^{n} d_k^{2-p}|c_k|^p)^{1/p}.$$

(This $\| \ \|_{d,l_p}$ is a norm only in a generalized sense of the word.)

Theorem 1: Suppose that $\{z|\,|z|<1\}$ lies inside L, and that some arc $\{e^{it}|\ a<t<b\}$ (where $0\leq a<b\leq 2\pi$) is part of L. Then, if $0<p<2$, $\lim_{n\to\infty} \|U_n\|_{d,l_p} = \infty$ if and only if $\sum_{k=0}^{\infty} d_k^2 = \infty$. It is pointed out that the hypothesis $p<2$ and the assumption that L contains some arc of the unit circle cannot be omitted.

Next, theorem 1 is used to deduce the existence of analytic functions satisfying certain conditions. Theorem 2: If $0<p<2$ and $\sum_{k=0}^{\infty} d_k^2=\infty$ ($d_k\geq 0$; $k=0, 1, 2, \cdots$), there exists a function $f(z)=\sum_{k=0}^{\infty} c_k z^k$ which is analytic everywhere except perhaps on the ray $[1, \infty)$, continuous on the disk $|z|\leq 1$, and such that $\sum_{k=0}^{\infty} d_k^{2-p}|c_k|^p=\infty$. Theorem 3: Let $t_0, t_1, \cdots, t_n, \cdots$ be a sequence of non-negative numbers such that $\sum_{n=0}^{\infty} t_n^r=\infty$ for all $r>0$. Then there exists a function $f(z)=\sum_{n=0}^{\infty} c_n z^n$ which is analytic everywhere except perhaps on the ray $[1, \infty)$, continuous on the disk $|z|\leq 1$, and such that $\sum_{n=0}^{\infty} t_k |c_k|^{2-\varepsilon}=\infty$ for all positive $\varepsilon<2$.

In his proofs the author makes considerable use of both classical and functional analysis.

J. M. G. Fell (Cambridge, Mass.)

2687:

Newman, Donald J. Insertion of ± signs in e^x. Proc. Amer. Math. Soc. **11** (1960), 444–446.

Let $\varepsilon_n=\pm 1$ denote arbitrary signs. A function $f(x)=\sum_{n=0}^{\infty} \varepsilon_n x^n/n!$ cannot satisfy $f(x)\to 0$ as $|x|\to+\infty$. It satisfies $f(x)=O(e^{\rho x})$, $\rho<1$ for $x\to+\infty$ if and only if the ε_n are periodic for large n with an even period $2K$ for which $\varepsilon_{n+1}+\varepsilon_{n+2}+\cdots+\varepsilon_{n+2K}=0$.

G. G. Lorentz (Syracuse, N.Y.)

2688:

Roux, Delfina. Una estensione del teorema di Fabry-Pólya-Ricci relativo al punto singolare delle serie di potenze. Ann. Mat. Pura Appl. (4) **47** (1959), 59–73. (English summary)

Author's summary: "Si stabilisce una condizione sufficiente affinchè il punto $z=1$ sia singolare per la serie $\sum a_n z^n$ (di raggio 1); tale condizione tiene conto della cosiddetta «sopra-emisimmetria» per le componenti dei coefficienti appartenenti ad una successione di tratti, consente eccezioni alla sopra-emisimmetria e consente inoltre la posizione eccentrica generica per la coppia di componenti a somma preponderante. Nella forma generale della condizione si inquadrano, come casi particolari, una buona parte delle proposizioni note sull'argomento: in particolare tutte quelle che tengono conto della sopra-emisimmetria."

F. Herzog (E. Lansing, Mich.)

2689:

Isahanov, R. S. Differential boundary problem of a linear union and its application in the theory of integro-differential equations. Soobšč. Akad. Nauk Gruzin. SSR **20** (1958), 659–666. (Russian)

The author studies the conditions for solvability and explains the methods for constructing the solutions of the following problems: (I) To find piecewise holomorphic

functions $\Phi(z)$ of finite order at infinity with preassigned boundary condition

$$(1)\quad \sum_{k=0}^{m}\left\{A_k(t)\Phi^{(k)}{}_+(t)+B_k(t)\Phi^{(k)}{}_-(t) + \frac{1}{\pi i}\int_L [R_k(t,\xi)\Phi^{(k)}{}_+(\xi)+S_k(t,\xi)\Phi^{(k)}{}_-(\xi)]\frac{d\xi}{\xi-t}\right\}=g(t),$$

where $t\in L$ is a simple closed smooth contour. (II) To find solutions of the integro-differential equation

$$(2)\quad \sum_{r=0}^{m}\left\{a_r(t)u^{(r)}(t)+\frac{1}{\pi i}\int_L \frac{K_r(t,\xi)u^{(r)}(\xi)d\xi}{\xi-t}\right\}=f(t)\quad (t\in L),$$

whose mth derivative $u^{(m)}(t)$ satisfies a Hölder condition. It is assumed that the given functions appearing in (1) and (2) satisfy a Hölder condition.

Ya. V. Bykov (RŽMat. **1960** #4146)

2690:

Isahanov, R. S. Some differential boundary problems in the theory of analytic functions. Soobšč. Akad. Nauk Gruzin. SSR **21** (1958), 11–18. (Russian)

The author investigates the following boundary problem, and the integro-differential equation with Cauchy kernel connected with it, for a multiply connected domain bounded by $\rho+1$ smooth curves:

$$(1)\quad \sum_{k=0}^{m}\left[A_k(t)\Phi^{(k)}{}_+(t)+\frac{1}{\pi i}\int_L\frac{R_k(t,\tau)}{\tau-t}\Phi^{(k)}{}_+(\tau)d\tau\right] +\sum\left[B_k(t)\Phi^{(k)}{}_-(t)+\frac{1}{\pi i}\int_L\frac{S_k(t,\tau)}{\tau-t}\Phi^{(k)}{}_-(\tau)d\tau\right]=g(t).$$

By means of integral representation of the desired piecewise analytic function $\Phi^{\pm}(z)$ the boundary problem is reduced to the singular integral equation

$$(2)\quad A(t)\mu(T)+\frac{1}{\pi i}\int\frac{K(t,\tau)}{\tau-t}\mu(\tau)d\tau = F(t)$$

with respect to the density $\mu(t)$ of the integral representation, and a set of supplementary conditions of the form

$$(3)\quad \int\mu(t)t^j dt = 0\quad (j=0,1,\cdots,m-1).$$

By a special method due to I. N. Vekua, the problem consisting of equation (2) and conditions (3) can be reduced to a single singular equation of form (2) with respect to a certain other auxiliary function. The author also considers the more general boundary problem with translation

$$(4)\quad \sum_{k=0}^{m}\left\{A_k(t)\Phi^{(k)}{}_+[\alpha_k(t)]+\frac{1}{\pi i}\int\frac{R_k(t,\tau)}{\tau-t}\Phi^{(k)}{}_+[\beta_k(\tau)]d\tau\right\} +\sum_{k=0}^{m}\left\{B_k(t)\Phi^{(k)}{}_-[\gamma_k(t)]+\frac{1}{\pi i}\int\frac{S_k(t,\tau)}{\tau-t}\Phi^{(k)}{}_-[\delta_k(\tau)]d\tau\right\}=g(t)$$

with the restriction

$$(5)\quad \alpha_m(t)=\beta_m(t),\quad \gamma_n(t)=\delta_n(t)=t.$$

The solution of this problem is reduced to a singular integral equation of type (2).

{Reviewer's remark. The author is unaware of the article of M. P. Ganin [Dokl. Akad. Nauk SSSR **79** (1951), 921–924; MR **13**, 223] and of his dissertation (Kazan, 1952), the results of the latter being presented also in a monograph of the reviewer's [*Kraevye zadači*, Gosudarstv. Izdat. Fiz. Mat. Lit., Moscow, 1958; MR **21** #2879; pp. 360–365], in which the boundary problem (1) is reduced directly, by means of a certain integral representation, to a singular integral equation (2) without supplementary conditions (3). In the same monograph the boundary problem (4) is also investigated without the supplementary conditions (5).}

F. D. Gahov (RŽMat **1959** #11006)

2691:

Nakai, Mitsuru. On a problem of Royden on quasiconformal equivalence of Riemann surfaces. Proc. Japan Acad. **36** (1960), 33–37.

Let R be a Riemann surface and HBD(R) the space of bounded harmonic functions on R which have a finite Dirichlet integral. Define a ring structure in HBD(R) by defining $u\times v$ to be the projection on HBD(R) of the pointwise product of u and v. The reviewer proved [Proc. Amer. Math. Soc. **5** (1954), 266–269; MR **15**, 695] that if R and R' are quasiconformally equivalent, then HBD(R) and HBD(R') are isomorphic. Moreover, the converse was conjectured there. The author shows that the conjecture is false by constructing two Riemann surfaces R and R' which are topologically but not quasi-conformally equivalent for which the rings HBD(R) and HBD(R') are isomorphic and homeomorphic.

The author defines a Riemann surface to be HBD maximal if there is no prolongation to which all HBD functions can be extended. He points out that the surfaces given in the example are not HBD maximal and that the conjecture is still open for HBD maximal surface.

A result of independent interest that is used in the paper is that any linear subspace of HB functions can have at most one multiplication defined on it which preserves upper bounds. *H. L. Royden* (Stanford, Calif.)

2692:

Kuramochi, Zenjiro. Cluster sets of analytic functions in open Riemann surfaces with regular metrics. I. Osaka Math. J. **11** (1959), 83–90.

Up to now, cluster sets have been studied mainly for functions meromorphic in a plane domain D with boundary Γ. The author here investigates cluster sets of functions meromorphic on a Riemann surface R which is imbedded in another Riemann surface R^*. {The reviewer has some difficulties in following the reasoning.}

F. Huckemann (Giessen)

2693:

Tamura, Jirô. Meromorphic functions on open Riemann surfaces. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **9** (1959), 175–186.

The author obtains analogues of the fundamental theorems of R. Nevanlinna in the theory of meromorphic functions for the case of meromorphic functions on a noncompact Riemann surface whose conformal universal covering is hyperbolic. The method employed is based on the use of the exhaustion of the fundamental domain Δ (center of Δ at 0) of a Fuchsian group acting on $\{|z|<1\}$: $(F_r)_{0<r<1}$, where $F_r=\Delta\cap\{|z|\le r\}$. An analogue of the second fundamental theorem is obtained by modifying the F_r somewhat and applying the generalization of a formula of Ahlfors given by Kunugui [Japan. J. Math. **18** (1942),

1–39; MR 8, 24] and Tumura [ibid. 18 (1942), 303–322; MR 7, 516]. *M. H. Heins* (Urbana, Ill.)

2694:

Bers, Lipman. Simultaneous uniformization. Bull. Amer. Math. Soc. **66** (1960), 94–97.

A group G of Möbius transformations will be called quasi-Fuchsian if there exists an oriented Jordan curve γ_G (on the Riemann sphere P) which is fixed under G, and if G is fixed-point-free and properly discontinuous in the domains $I(\gamma_G)$ and $E(\gamma_G)$ interior and exterior to γ_G, respectively. If γ_G is a circle, G is a Fuchsian group.

Let S be an abstract Riemann surface, f a quasi-conformal map onto another such surface S' with $[f]$ the homotopy class of f. We call $(S, [f], S')$ a coupled pair of Riemann surfaces. Consider now a fixed closed Riemann surface S_0 of genus $g>1$. The equivalence classes of coupled pairs $(S, [f], S_0)$ are the points of the Teichmuller space T_g. It is known that T_g has a natural complex-analytic structure and can be represented as a bounded domain in the number space C^{3g-3}; also T_g is homeomorphic to a cell [cf. following review]. If $\tau=(\tau_1, \cdots, \tau_{3g-3}) \in T_g$, we denote by $(S_\tau, [f_\tau], S_0)$ any pair represented by τ.

Theorem 2: There exist $2g$ Möbius transformations $A_j^{(\tau)}$ which depend holomorphically on $\tau \in T_g$, satisfy normalization conditions, and generate, for each fixed τ, a quasi-Fuchsian group G_τ with $I(\gamma_{G_\tau})/G_\tau$ conformally equivalent to S_τ.

Let $T_{g,1}$ be the Teichmuller space of compact Riemann surfaces with one point distinguished. Theorem 3: $T_{g,1}$ is holomorphically equivalent to the domain $M_{g,1} \subset C^{3g-2}$ defined as follows: $(z, \tau)=(z, \tau_1, \cdots, \tau_{3g-3}) \in M_{g,1}$ if and only if $\tau \in T_g$ and $z \in I(\gamma_{G_\tau})$.

Theorem 4: There exist finitely many meromorphic functions, $F_1(z, \tau), \cdots, F_N(z, \tau)$ in $M_{g,1}$ which, for every fixed τ, generate the field of automorphic functions in $I(\gamma_{G_\tau})$ under the group G_τ, i.e., the field of meromorphic functions on S_τ. *H. L. Royden* (Stanford, Calif.)

2695:

Bers, Lipman. Spaces of Riemann surfaces as bounded domains. Bull. Amer. Math. Soc. **66** (1960), 98–103.

Let $T_{g,n}$ be the Teichmuller space of all "marked" compact Riemann surfaces of genus g with n distinguished points. Then $T_{g,n}$ is a complex $(3g-3+n)$-dimensional manifold. The author proves that $T_{g,n}$ is holomorphically equivalent to a bounded domain in Euclidean space. *H. L. Royden* (Stanford, Calif.)

2696:

Akaza, Tohru. A property of some Poincaré theta-series. Nagoya Math. J. **16** (1960), 189–194.

Soit Γ le groupe fuchsien, engendré par les transformations hyperboliques à coefficients réels $S_\nu(z)$, dont le domaine fondamental est limité par des cercles disjoints c_ν $(\nu=\pm 1, \pm 2, \cdots)$ s'accumulant à l'infini des deux côtés de l'axe réel; et soit

$$\chi_{ab}(z) = \sum_{S \in \Gamma} \log \frac{S(z)-a}{S(z)-b} : \frac{S(0)-a}{S(0)-b}.$$

L'A. étudie le type de $\chi_{ab}(z)$, et obtient des résultats analogues à ceux de L. Myrberg [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. No. 71 (1950); MR 12, 90]. *J. Lelong* (Paris)

2697:

Kaplan, Wilfred. Paths of rapid growth of entire functions. Comment. Math. Helv. **34** (1960), 71–74.

A. Huber [Comment. Math. Helv. **32** (1957), 13–72; MR **20** #970] proved that if $f(z)$ be an entire function, not a polynomial, and if $\lambda>0$, then there exists a locally rectifiable path C_λ tending to ∞ such that $\int_{C_\lambda} |f(z)|^{-\lambda}|dz| < \infty$. Huber's proof depends on properties of subharmonic functions. The author gives a simple proof of this theorem when $\lambda \geq 1$ or $\lambda=1-(1/n)$ $(n=2, 3, \cdots)$, and remarks that this theorem can be improved if $f(z)$ has no zeros.

In the paper referred to above, Huber raised the question as to whether (*) $\int_1^\infty |f(re^{i\theta})|^{-\lambda}dr=\infty$ for all θ, $0 \leq \theta < 2\pi$, $\lambda>0$, implies that $f(z)$ is a polynomial. The author answers this question. He shows, with the help of a theorem of Keldyš and Mergelyan, that there exists an entire function $f(z)$ such that each half-line in the z-plane contains infinitely many disjoint segments of length 1 on which $|f(z)|<1$. Hence there exist transcendental entire functions $f(z)$ satisfying (*) for all θ, $0 \leq \theta < 2\pi$. [See the succeeding review.] *S. M. Shah* (Lawrence, Kansas)

2698:

Piranian, George. An entire function of restricted growth. Comment. Math. Helv. **33** (1959), 322–324.

The author proves that there exists a sequence $\{(t_n, r_n)\}$ such that the function $f(z)=\prod_1^\infty \{1-(z/r_n)^n\}^{t_n}$ is entire and has the property that each half-line in the z-plane contains infinitely many disjoint segments of length 1 and on which $|f(z)|<1$ [see the preceding review]. Furthermore, given any real-valued function $h(r)$ such that $h(r)/\log r \to \infty$ as $r \to \infty$, the sequence $\{(t_n, r_n)\}$ can be chosen in such a way that $\log |f(re^{i\theta})| < h(r)\log r$, for all r $(r>r_0)$ and all θ. This function has zeros at points $r_n \exp(2h\pi i/n)$ $(h=1, 2, \cdots, n;\ n=1, 2, \cdots)$, and the author chooses the pairs (t_n, r_n) in such a way that $r_n>n^3$ and $|f(z)|<1/n$ throughout each of the annular sectors $s_{n,h}$ $(h=1, 2, \cdots, n;\ n=1, 2, \cdots)$ determined by the conditions $z=(1+s)r_n \exp(i(\theta+2\pi h/n))$ $(s, \theta$ real; $|s|<1/n^2$, $|\theta|<\pi/n^2)$. [See the succeeding review.] *S. M. Shah* (Lawrence, Kansas)

2699:

Hayman, W. K. Slowly growing integral and subharmonic functions. Comment. Math. Helv. **34** (1960), 75–84.

Denote by an $\mathscr{E}$-set any countable set of circles not containing the origin and subtending angles at the origin whose sum is finite. Let $u(z)$ be subharmonic and not constant in the plane, and write $B(r)=B(r, u)=\sup_{|z|=r} u(z)$. The author proves that if $B(r, u)=O((\log r)^2)$ as $r \to \infty$, then $u(re^{i\theta}) \sim B(r)$ uniformly as $re^{i\theta} \to \infty$ outside an $\mathscr{E}$-set. If $B(r)=O((\log r))$, as $r \to \infty$, then $u(re^{i\theta})=B(r)+o(1)$ uniformly as $re^{i\theta} \to \infty$ outside an $\mathscr{E}$-set.

Taking $u(z)=\log |f(z)|$, where $f(z)$ is an entire function satisfying (*) $\log M(r, f)=O((\log r)^2)$, and $M(r, f)$ denotes as usual the maximum modulus of $f(z)$, it follows that (**) $\log |f(re^{i\theta})| \sim \log M(r, f)$, uniformly as $re^{i\theta} \to \infty$, outside an $\mathscr{E}$-set. Valiron proved that (**) holds outside a set of linear density 0. Since an $\mathscr{E}$-set has linear density 0,

the author's result (**) is stronger than that of Valiron. It also shows that if an entire function $f(z)$ satisfies (*), then for almost every fixed θ, $|f(re^{i\theta})|\to\infty$ as $r\to\infty$, in contrast to the behaviour of the function constructed by Piranian [see the preceding review].

S. M. Shah (Lawrence, Kansas)

2700:

Lohin, I. F. On linear aggregates of entire functions. Mat. Sb. (N.S.) **49 (91)** (1959), 341–346. (Russian)

The author proves a generalized Liouville theorem whose statement involves the relative type of entire functions. Let $f(z)=\sum a_n z^n$ be a comparison (entire) function, that is, $a_n\neq 0$ for all n and

$$\limsup \Big\{\sum_{k=0}^{n} |a_k a_{n-k}/a_n|\Big\}^{1/n} < \infty.$$

Now let $L(z)=\sum c_n z^n$ be an arbitrary entire function of finite relative type $\limsup |c_n/a_n|^{1/n}$, with zeros λ_ν of multiplicities k_ν. Suppose that the function Φ of finite relative type can be written as the limit, uniformly in every bounded domain, of a sequence of linear combinations

$$P_n(z) = \sum_{\nu=1}^{p_n} \{a_{\nu 1}^{(n)} f(\lambda_\nu z) + \cdots + a_{\nu k_\nu}^{(n)} z^{k_\nu-1} f^{(k_\nu-1)}(\lambda_\nu z)\}.$$

Then Φ is equal to a finite linear combination of the form of P_n. The proof is similar to that of an analogous theorem involving ordinary type which was obtained by Leont'ev [Mat. Sb. (N.S.) **31 (73)** (1952), 201–208; MR **14**, 154].

J. Korevaar (Madison, Wis.)

2701:

Srivastava, K. N. On the means of an entire function and its derivatives. Quart. J. Math. Oxford Ser. (2) **10** (1959), 230–232.

Let f be an entire function, and define, for any $\delta\geq 1$,

$$w(r,f) = \log \mu_\delta(r,f) = \log\Big\{(2\pi)^{-1}\int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta\Big\}.$$

It is known that the order of f can be computed as $\rho=\limsup_{r\to\infty} \log w(r,f)/\log r$. It is also known that f and f' have the same orders. The author shows by an ingenious argument that

$$\rho = 1+\limsup_{r\to\infty} \frac{w(r,f')-w(r,f)}{\delta \log r}$$

Moreover, the same formula yields the lower order λ of f if lim sup is replaced by lim inf.

R. C. Buck (Madison, Wis.)

2702:

Lohwater, A. J. The exceptional values of meromorphic functions. Colloq. Math. **7** (1959), 89–93.

A descriptive lecture containing an announcement, without proof, of a result relating certain cluster sets, a range of values, and asymptotic values, of a function meromorphic in the unit disk.

F. Bagemihl (Ann Arbor, Mich.)

2703:

Makmak, K. M. On certain properties of functions with limited boundary rotation. II. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 1054–1059. (Ukrainian. Russian and English summaries)

[For part I see same Dopovidi **1959**, 567–570; MR **21** #3572.] Author's summary: "Three extremal problems are solved in this paper: the curvature fluctuations are estimated, the value of $|f'(z)|+|f'(-z)|$ and the length of the circumference contour are determined for certain functions."

2704:

Šlionskiĭ, G. G. On the theory of bounded schlicht functions. Vestnik Leningrad. Univ. **14** (1959), no. 13, 42–51. (Russian. English summary)

Proofs of results announced earlier by the author in Dokl. Akad. Nauk SSSR **111** (1956), 962–964 [MR **18**, 798].

A. W. Goodman (Lexington, Ky.)

2705:

Dundučenko, L. O.; Kas'yanyuk, S. A. On analytic functions of limited boundary rotation in n-connected circular regions. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 227–230. (Ukrainian. Russian and English summaries)

Author's summary: "Proceeding from the results of V. A. Zmorovič [same Dopovidi **1958**, 489–492; MR **20** #5277], the authors introduce a class $P(K_n)$ of functions with limited boundary rotation which generalizes the corresponding class of functions of V. Paatero [Ann. Acad. Sci. Fenn. Ser. A. I. Math.-Phys. No. 147 (1953); MR **15**, 33] on n-connected circular regions K_n. The structural formula of this class of functions is established and exact estimates are found for the expressions $\arg f'(z)$ and $|f'(z)|$ from which, as special cases, follow the results: with $n=2$ that of the authors; with $n=1$, that of V. Paatero [loc. cit.]. An n-connected analogue was obtained incidentally for Schwarz-Christoffel's well-known polygonal formula in conformal mapping theory."

2706:

Brown, R. K. Univalence of Bessel functions. Proc. Amer. Math. Soc. **11** (1960), 278–283.

The author shows that the normalized Bessel function $[J_\nu(z)]^{1/\nu}$ is regular, univalent and spiral-like in every circle $|z|=r<\rho_\mu$, where $\mu^2=\mathscr{R}(\nu^2)$, $\mu>0$, $|\arg \nu|<\frac{1}{4}\pi$, and ρ_μ is the smallest positive zero of the function $J_\mu'(r)$. When ν is real and positive the function $[J_\nu(z)]^{1/\nu}$ is star-like in $|z|<\rho_\mu$ but is not univalent in any larger circle.

If $\nu=x+iy$ satisfies either (1) $0\leq x<1$ and $y\leq x$ or (2) $x\geq 1$ and $y^2<2x-1$, then the normalized Bessel function $z^{1-\nu}J_\nu(z)$ is regular, univalent and spiral-like in every circle $|z|=r<\rho_\mu^*$, where $\mu^2=\mathscr{R}(\nu^2)$, $\mu>0$, and ρ_μ^* is the smallest positive zero of the function $rJ_\mu'(r)+\mathscr{R}(1-\nu)J_\mu(r)$. If ν is real ρ_μ^* is the radius of univalency of $z^{1-\nu}J_\nu(z)$ and in this case this function is star-like in $|z|<\rho_\mu^*$.

The method of proof depends on a knowledge of the univalence of functions $[W(z)]^{1/\alpha}$, $\mathscr{R}\alpha\geq\frac{1}{2}$, where $W(z)$ is a solution of the differential equation $W''(z)+\phi(z)W(z)=0$, and α is a root of the corresponding indicial equation; and in particular upon theorem A of the reviewer's paper, Trans. Amer. Math. Soc. **76** (1954), 254–274 [MR **15**, 786].

M. S. Robertson (New Brunswick, N.J.)

2707:

Tammi, Olli. On bounded univalent functions. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 413–417. (Russian summary, unbound insert)

Let

$$f(z, x) = z + \sum_{n=2}^{\infty} a_n(x) z^n \quad (0 \leq x \leq 1)$$

be univalent and $|f(z, x)| < x^{-1} = R$ for $|z| < 1$. The author continues his studies of extremal properties of these functions [Ann. Acad. Sci. Fenn. Ser. A. I. No. 149 (1953); no. 162 (1953); MR **15**, 302, 516].

Let L be a fixed schlicht domain. Consider the class of all normalized functions $f(z) = z + a_2 z^2 + \cdots$ which map the unit circle $|z| < 1$ onto domains D lying inside L; such domains D are called normalized. Denote the totality of such domains by $\{D\}_L$ and the corresponding mapping functions by $\{f(z)\}_L$. Let $w = L(w')$, $L(0) = 0$, $L'(0) = 1$ map the circle K_ρ: $|w'| \leq \rho$ $(\rho > 1)$ onto the given domain L. Let D' be the image of D under the mapping $w' = L^{-1}(w)$. Since both functions are normalized, the function $w' = L^{-1}(f(z)) = g(z)$ maps the unit circle onto D' and is normalized; and conversely if D' is normalized in $|w'| \leq \rho$ then $f(z) = L(g(z))$ is normalized and $D \in \{D\}_L$. Hence there is a 1-1 correspondence between the classes $\{D'\}_{K_\rho}$; $\{D\}_L$ and $\{g(z)\}_{K_\rho}$; $\{f(z)\}_L$, respectively, with the corresponding members being related by $w = L(w')$. The author obtains the Koebe constant for $\{D\}_L$. He compares the extremal problem $|a_2| = \max.$ of the class $\{g\}_{K_\rho}$ with the class $\{f\}_L$ in the special case where L is the circle $|w - c| \leq R$, and using results from previous works cited above discusses the nature of the extremal domains for $a_3(x)$, noting that the circle $|w - c| \leq R$ is a more convenient bounding domain for D than the circle $|w| \leq R$.

W. C. Royster (Lexington, Ky.)

2708:

Dundučenko, L. O.; Kas'yanyuk, S. A. On classes of functions of limited form in n-connected circular regions. Dopovidi Akad. Nauk Ukrain. RSR **1959**, 945–948. (Ukrainian. Russian and English summaries)

Authors' summary: "This note discusses functions regular in K_n of a limited form of classes A and H_p $(p > 0)$. A structural formula of class A is established which generalizes V. I. Smirnov's well-known result for n-connected circular regions K_n."

2709:

Akutowicz, Edwin J. Schwarz's lemma in the Hardy class H^1. Rend. Circ. Mat. Palermo (2) **8** (1959), 185–191.

Suppose that two sequences of complex numbers z_n, w_n $(n = 1, 2, 3, \cdots)$ are given, the w_n not all zero, $|z_n| < 1$, $z_n \neq z_m$ if $n \neq m$, and z_n without limit points in $|z| < 1$, such that there is more than one function φ in the Hardy class H^1 satisfying the interpolation condition $\varphi(z_n) = w_n$ $(n = 1, 2, \cdots)$. Let f denote an extremal function in H^1 for the problem $\inf \|\varphi\|$ where φ satisfies the above interpolation condition and

$$\|\varphi\| = \lim_{r \to 1} (2\pi)^{-1} \int_0^{2\pi} |\varphi(re^{it})| dt.$$

Let B be the Blaschke product with zeros at z_n $(n = 1, 2, \cdots)$. If F denotes the holomorphic function in $|z| < 1$ satisfying $|F(z)| \leq 1$ in $|z| < 1$ which coincides almost everywhere on $|z| = 1$ with $B|f|/zf$, then F is an extremal function for the conjugate problem $\sup |\lambda(\Phi)|$, where $\lambda(\Phi) = (2\pi i)^{-1} \int_{|z|=1} f\Phi/B\, dz$ and the supremum is taken over all holomorphic functions Φ in $|z| < 1$ satisfying $|\Phi(z)| \leq 1$ there. Let E denote the set of limit points of the sequence z_n on $|z| = 1$ and assume that E does not coincide with the whole periphery. The author's main theorem states that every pair of conjugate extremal functions f and F can be continued analytically as meromorphic functions into the complement of the set E in the finite plane.

G. Springer (Lawrence, Kans.)

2710:

Erwe, Friedhelm. Abschätzungen für beschränkte Funktionensysteme. Math. Z. **71** (1959), 414–426.

On considère un système $f(z) = [f_1(z), \cdots, f_n(z)]$ de n fonctions holomorphes de la variable complexe z dans le cercle $|z| < 1$; $f(z)$ est écrit comme matrice colonne; $f \to f^*$ est le passage à la transposée conjuguée; on pose $|f|^2 = \sum |f_i|^2$, $v(a) = \sqrt{(1 - |a|^2)}$. On généralise alors des résultats connus pour $n = 1$: (I) $f(z)$ et $g(z)$ vérifiant $|f(z)| < 1$, $|g(z)| < 1$ pour $|z| < 1$, ainsi que $f(0) = a$, $g(0) = b$, on établit une majoration et une minoration de $|f(z) - g(z)|$. (II) Si, de plus, $a \neq b$, on donne une expression explicite $R(a, b)$, telle que $f(z_1) \neq g(z_2)$ pour $|z_1| < R(a, b)$, $|z_2| < R(a, b)$. (III) On pose $\Gamma(a) = (1 + v(a))^{-1} aa^* + v(a)E$, E étant la matrice unité. Une rotation de la sphère $|z| < 1$ de C^n est définie alors par $Z \to \Gamma(a)(Z - a)/(1 - a^* Z)$ dans C^n. On établit une majoration

$$\left| \Gamma(g(z)) \frac{f(z) - g(z)}{1 - g^*(z) f(z)} \right| \leq \psi[z, R(a, b)]$$

où ψ est donnée explicitement. En particulier on a, les limitations étant précises:

$$|f(z) - a| \leq [1 - |a|^2] \frac{|z|}{1 - |a|\,|z|}$$

pour $n = 1$, et pour $n > 1$, $|z| \geq |a|$; tandis qu'on a $|f(z) - a| \leq (1 - |a|^2)^{1/2} |z| / (1 - |z|^2)$ pour $n > 1$, $|z| \leq |a|$.

P. Lelong (Paris)

2711:

Kas'yanyuk, S. A. On some classes of functions bounded in an annulus. Ukrain. Mat. Ž. **11** (1959), 52–65. (Russian. English summary)

The author establishes a number of sharp inequalities satisfied by the members of various classes of functions which are analytic on an annulus (such as the class of analytic functions with modulus at most one, with positive real part, with real part bounded in modulus by one, etc.). A detailed study is made of the class $\mathfrak{A}_L$ consisting of the non-vanishing analytic functions f on a given annulus $\{q < |z| < 1\}$ which satisfy

$$\oint_{|z|=p} \frac{f'}{f} dz = 0, \quad \oint_{|z|=p} \frac{\log f}{z} dz = 0, \quad q < p < 1,$$

as well as the condition that

$$L_f = (2\pi)^{-1} \int_\Gamma \log |f(z)|\, d \arg z$$

exists in a limiting sense, Γ being the boundary of $\{q<|z|<1\}$ taken in the positive sense. Here sharp inequalities are obtained for $|f|$, $f\in\mathfrak{A}_L$.

M. H. Heins (Urbana, Ill.)

2712:

Gol'dberg, A. A. Notes on Blaschke's derivatives for a half-plane. Ukrain. Mat. Ž. **11** (1959), 210–213. (Russian)

Let $a_k=r_k\exp(i\theta_k)$ be a sequence of complex numbers such that $0<\theta_k<\pi$, $r_k\to\infty$, $\sum_{k=1}^{\infty}\sin\theta_k/r_k<+\infty$. Then

$$f(z)=\prod_{k=1}^{\infty}\frac{1-\bar{a}_k^{-1}z}{1-a_k^{-1}z}$$

is convergent for $z\neq a_k$ or $\bar{a}_k$, and represents a meromorphic function in the plane, such that $|f(z)|>1$ for $y>0$, where $z=x+iy$. Set

$$m(r,f)=(2\pi)^{-1}\int_0^{\pi}\log^+|f(r\exp(i\phi))|d\phi.$$

It follows from a theorem of the reviewer [J. Math. Pures Appl. (9) **35** (1956), 115–126; MR **17**, 1073] that (1) $m(r,f)=o(r)$ as $r\to\infty$ outside a set of finite logarithmic measure. The author now shows that (a) (1) holds without exceptional r if

$$\sum_{k=1}^{\infty}\frac{\sin\theta_k}{r_k}\log\frac{2}{\sin\theta_k}<+\infty,$$

or if $f(z)$ has order 1 mean type at most, but not always for $f(z)$ having order 1, maximal type; (b) if $f(z)$ has finite order $m(r)=o(r\log r)$.

W. K. Hayman (London)

2713:

Mikhail, M. N. The effectiveness of the basic sets of polynomials at a point. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 480–484.

In a previous paper [Duke Math. J. **20** (1953), 459–479; MR **15**, 23], the author associated with any basic set $(p_n(z))$ three (extended-real) numbers b, B, C, and in terms of these he characterized the annulus of effectiveness for basic sets satisfying $D_n=O(n)$. (Here D_n denotes, as usual, the degree of the polynomial of highest degree appearing in the representation $z^n=\sum\pi_{nk}p_k(z)$.) The three theorems stated in the present paper are the converses of those intended. The first should read: If (i) C is finite, (ii) $b=0$ and (iii) $D_n/n\to1$, then the set is effective at the origin. The author does not mention condition (iii), claiming that his result applies to any basic set. However, the result becomes false if (iii) is even weakened to $D_n=O(n)$.

The other two intended theorems are special cases of results of the reviewer [Philos. Trans. Roy. Soc. London Ser. A **245** (1953), 429–468; MR **14**, 968], and the proofs offered are invalid.

{Editor's note: The author acknowledges that "Newns's results in this connection imply the results of this paper. The aim, here, was to test the validity of the author's constants to reach the conclusions."}

W. F. Newns (Liverpool)

2714:

Kulshrestha, P. K. On a question of representation of analytic functions by Faber polynomials. Bull. Calcutta Math. Soc. **51** (1959), 73–76.

The author shows that the Faber expansion of an analytic function can be summed by Mittag-Leffler's method in the Mittag-Leffler star.

[This problem was considered earlier by Ullman, Michigan Math. J. **2** (1954), 109–114; MR **16**, 121; and Lohin Mat. Sb. (N.S.) **36** (**78**) (1955), 441–444; MR **16**, 1093.]

R. P. Boas, Jr. (Evanston, Ill.)

2715:

Walsh, J. L. On the asymptotic properties of extremal polynomials with prescribed constant term. Math. Z. **73** (1960), 339–345.

E désignant un ensemble plan fermé, de diamètre transfini $\tau(E)>0$, l'A. étudie les polynômes $F_n(z)=z^n+a_1z^{n-1}+\cdots+a_{n-1}z+A_n$ dont le coefficient $A_n=F_n(0)$ est donné, et de norme $\|F_n\|=\sup_{z\in E}|F_n(z)|$ minimal. Il établit d'abord une condition nécessaire et suffisante (différente selon que E contient ou non l'origine) que doivent vérifier les A_n pour que l'on ait $\lim\|F_n\|^{1/n}=\tau(E)$; puis il étudie les propriétés asymptotiques des $F_n(z)$ découlant de la répartition de leurs zéros.

J. Lelong (Paris)

2716:

Orudžev, G. A. On convergence of an interpolation series of rational fractions. Izv. Akad. Nauk Azerbaĭdžan. SSR. Ser. Fiz.-Teh. Him. Nauk **1958**, no. 4, 3–22. (Russian. Azerbaijani summary)

The convergence of series of the form

$$(1)\qquad \sum_{n=1}^{\infty}a_n\frac{(1-\alpha_1^{-1}z)(1-\alpha_2^{-1}z)\cdots(1-\alpha_n^{-1}z)}{(1-\beta_1^{-1}z)(1-\beta_2^{-1}z)\cdots(1-\beta_n^{-1}z)}$$

is studied. A typical theorem is the following. Let α_k and β_k be a sequence of complex numbers for which $\lim_{k\to\infty}\alpha_k=\lim_{k\to\infty}\beta_k=\infty$, $\sum_{k=1}^{\infty}|\alpha_k^{-1}-\beta_k^{-1}|=\infty$, $|\arg(\alpha_k^{-1}-\beta_k^{-1})e^{i\alpha}|\leqq\eta_1<\frac{1}{2}\pi$, $0\leqq\alpha<2\pi$. Then, the series (1) converges absolutely in the interior of the region determined by the condition

$$|\arg[(z-Le^{i\alpha})(\alpha_k^{-1}-\beta_k^{-1})]|\leqq\eta<\tfrac{1}{2}\pi\qquad(k=n_0,n_0+1,\cdots),$$

where

$$L=\limsup_{n\to\infty}\frac{\log(|\alpha_n(\alpha_k^{-1}-\beta_k^{-1})^{-1}|)}{\operatorname{Re}\sum_{k=1}^{n}(\alpha_k^{-1}-\beta_k^{-1})e^{i\alpha}}.$$

In a final section, necessary conditions and sufficient conditions in terms of growth are derived for a function to possess an expansion in a Newton series with nodes $x_1,x_2,\cdots$, $x_1\leqq x_2\leqq\cdots$, $\lim_{n\to\infty}x_n=\infty$, $\sum_{k=1}^{\infty}1/x_k=\infty$.

{It may be useful to point out some special cases of (1) already in the literature: $1/\beta_i=0$ (Newton series) [see, e.g., A. O. Gel'fond, *Isčislenie konečnyh raznosteĭ*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952; MR **14**, 759]; $1/\alpha_i=0$ (factorial series) [see, e.g., Nörlund, *Vorlesungen über Differenzenrechung*, J. Springer, Berlin, 1924]; $\alpha_k=a+ku$, $\beta_k=b+kv$ [R. Lagrange, Acta Math. **64** (1935), 1–80]; $\beta_i=1/\alpha_i$, α_i real, monotonic increasing, $\lim_{i\to\infty}\alpha_i=1$, $\sum_{i=1}^{\infty}(1-\alpha_i)<\infty$ [V. L. and M. K. Gontčarov, Moskov. Gos. Univ. Uč. Zap. Matematika **73** (1944), 3–22; MR **7**, 202]; $\beta_i=1/\overline{\alpha_i}$, $|\alpha_n|\leqq|\alpha_{n+1}|$, $\lim_{n\to\infty}\alpha_n=\alpha$, $|\alpha|=1$, $\sum_{n=1}^{\infty}(1-|\alpha_n|)<\infty$ [J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc., New York, 1935].}

P. J. Davis (Washington, D.C.)

2717:

Bergman, Stefan. The number of intersection points of two analytic surfaces in the space of two complex variables. Math. Z. **72** (1959/60), 294–306.

Let $f(z_1, z_2)$ and $g(z_1, z_2)$ be analytic in a domain $\mathfrak{M}$ whose boundary consists of two analytic hypersurfaces. The author establishes an integral, extended over the "distinguished boundary surface" formed by the intersection of the aforementioned hypersurfaces, which furnishes the number of points in $\mathfrak{M}$ at which f and g simultaneously vanish. The author points out that if f and g are taken as the first derivatives of a function $p(z_1, z_2)$, the formula furnishes the number of critical points of p. It is then briefly indicated how the scope of the Morse-Heins theory of critical points may be extended from one to several complex variables.

Bernard Epstein (New York)

2718:

Scheja, Günter. Verzweigte Holomorphiehüllen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1958**, 9–18.

Par domaine (X, φ) au-dessus d'un espace analytique complexe Y, on entend un domaine qui est localement un recouvrement analytique ramifié de Y, φ étant la projection de Y sur X; une relation d'équivalence R entre points de (X, φ) qui est plus fine que $\varphi(x_1) = \varphi(x_2)$ est dite une contraction si $(X/R, \varphi/R)$ est encore un domaine au dessus de Y (les contractions correspondent aux décompositions continues analytiques les plus simples [cf. K. Stein, Math. Ann. **132** (1956), 63–93; MR **18**, 649]). A l'aide de cette notion, l'auteur étudie l'enveloppe d'holomorphie $\mathfrak{H}_F(X, \varphi) = (\tilde{X}, \tilde{\varphi})$ d'une famille F d'applications de (X, φ) dans un espace complexe Z; $(\tilde{X}, \tilde{\varphi})$ est défini par les propriétés suivantes: (I) il existe une application $\sigma | X \rightarrow \tilde{X}$ selon laquelle chaque élément de F se prolonge à $\tilde{X}$; (II) si (X', φ') est un domaine muni d'une application $\sigma' | \tilde{X} \rightarrow X'$, telle que chaque élément de F se prolonge selon $\sigma' \circ \sigma$ à X', σ' est bijective. On établit l'existence et l'unicité de l'enveloppe d'holomorphie $(\tilde{X}, \tilde{\varphi})_e$; σ est déterminée par les données (X, φ) et F. Un domaine holomorphe convexe (X, φ) au-dessus de C^n est domaine d'holomorphie; tout domaine faiblement convexe au-dessus de C^n est un domaine d'holomorphie.

P. Lelong (Paris)

2719:

Kerner, Hans. Holomorphiehüllen zu K-vollständigen komplexen Räumen. Math. Ann. **138** (1959), 316–328.

In 1932, P. Thullen showed that for every unramified Riemannian domain (X, Φ, C^n), one can associate its envelope of holomorphy (Holomorphiehülle) $\hat{H}(X, \Phi, C^n)$, which is essentially the smallest domain of holomorphy containing X. Later H. Cartan proved [Séminaire H. Cartan, École Norm. Sup., 1951–52; MR **16**, 233; VII] that the envelope of holomorphy for an unramified domain is invariant under biholomorphic transformations.

In the present paper, the author first proves in § 1 that one can construct the envelope of holomorphy $\hat{H}(X, \Phi, C^n)$ also for a ramified Riemannian domain (X, Φ, C^n) (Satz 2). This is called the Φ-Hülle of X. The same result was proved also by R. Iwahashi [J. Math. Soc. Japan **9** (1957), 452–463; MR **21** #2059] and by G. Scheja [see the preceding review]. However, contrary to the unramified case, the theorem of H. Cartan no longer holds. The Φ-Hülle is not invariant under biholomorphic transformations, but depends on the choice of the projection Φ spreading the domain X over C^n. Therefore it is necessary to construct another type of envelope invariant under biholomorphic transformations.

In § 2 of the present paper, the author introduces a new kind of envelope with such character called a K-Hülle. Here a few terminologies should be defined. A mapping φ: $R_1 \rightarrow R_2$ from a connected topological space R_1 into another topological space R_2 is said to be nowhere degenerate if, for every point $p_2 \in R_2$, the inverse image $\varphi^{-1}(p_2) \subset R_1$ consists of at most isolated points, and if the image $\varphi(R_1)$ contains an open set of R_2. A complex analytic space X is called K-complete if, for every point $x \in X$, there exist a finite number of functions $f_1, \cdots, f_n$ holomorphic over X such that the mapping f: $X \rightarrow C^n$ generated by the system of functions $(f_1, \cdots, f_n)$ is nowhere degenerate in a neighborhood of the point x. Finally, $\mathfrak{R}(X)$ means the ring consisting of all holomorphic functions over a complex analytic space X. Now, a complex analytic space $H(X)$ is called the K-Hülle of a complex analytic space X, if the following two conditions are satisfied: (1) There exists a holomorphic mapping α: $X \rightarrow H(X)$ nowhere degenerate on X such that the natural homomorphism $^*\alpha$: $\mathfrak{R}(H(X)) \rightarrow \mathfrak{R}(X)$ generated by α is an isomorphism between these two rings. (2) If there exists a K-complete complex analytic space Y with a nowhere degenerate mapping β: $X \rightarrow Y$, where $^*\beta$: $\mathfrak{R}(Y) \rightarrow \mathfrak{R}(X)$ is an isomorphism, then there is a nowhere degenerate holomorphic mapping γ: $Y \rightarrow H(X)$ with the property $\alpha = \gamma \circ \beta$. The author proves the existence and the uniqueness up to analytic isomorphisms of the K-Hülle $H(X)$ for every K-complete complex analytic space (Satz 3). In this case, $H(X)$ itself is again K-complete.

In § 3, the author considers the relations between the Φ-Hülle and the K-Hülle. For a Riemannian domain (X, Φ, C^n) with a K-complete complex analytic space X, the Φ-Hülle $\hat{H}(X, \Phi, C^n)$ is a subdomain of the K-Hülle $H(X)$, and the set $H(X) - \hat{H}(X, \Phi, C^n)$ coincides exactly with the degenerating set of the mapping Φ extended canonically to $H(X)$ (Satz 4). Several results follow from this theorem. For example, if $\hat{H}(X, \Phi, C^n) \neq H(X)$, then $\hat{H}(X, \Phi, C^n)$ is not holomorphic-convex (Satz 6). Finally, for an unramified Riemannian domain, its Φ-Hülle and K-Hülle are the same, which coincides also with the classical envelope of holomorphy in the sense of P. Thullen (Satz 8).

S. Hitotumatu (Tokyo)

SPECIAL FUNCTIONS

See also 2706, 2773, 2859.

2720:

Đoković, D. Sur une formule analogue à celle de décomposition des fractions rationnelles en éléments simples. Bull. Soc. Math. Phys. Macédoine **8** (1957), 41–43. (Serbo-Croatian. French summary)

Proof by induction of the identity

$$\sin^{n-1} x / \prod_{\nu=1}^{n} \sin(x - a_\nu) = \sum_{\nu=1}^{n} A_\nu \sin(x - a_\nu),$$

where $A_\nu = \sin^{n-1} a_\nu / \prod_{i \neq \nu} \sin(a_\nu - a_i)$. A geometrical proof was given by N. A. Kolmogorov, Kirov. Gos. Ped. Inst. Uč. Zap. **1953**, no. 7, 15–28 [MR **17**, 997].

2721:

Shelupsky, David. A generalization of the trigonometric functions. Amer. Math. Monthly **66** (1959), 879–884.

The author defines functions $\alpha_s(x)$ and $\beta_s(x)$, for any positive integral s, and proves by simple methods of classical analysis interesting generalisations of similar properties of $\sin x$ and $\cos x$. {The natural questions that arise in the mind of the reader are: (1) What are the generalisations of formulae like the addition formulae for $\sin x$ and $\cos x$ for these functions? (2) What are the power series for the two functions and do they define entire functions when $s>2$? Probably they are not entire functions since they are hyper-elliptic.} The values of $\Gamma(\frac{1}{4})$, $\Gamma(\frac{1}{5})$, $\cdots$ are given in terms of new numbers which the author defines as Π_s for various values of s; these numbers are nothing but four times the area bounded by the curves $x^s+y^s=1$ in the first quadrant. These functions may find a natural application in Minkowski geometry.

B. Mohan (Benares)

2722:

Carlitz, Leonard. Some integral equations satisfied by theta functions. Boll. Un. Mat. Ital. (3) **14** (1959), 489–492.

The author establishes the relation

$$\int_0^1 \frac{\theta_3(\phi, q)}{\theta_3(\phi+\omega, q)}\, d\phi = \tfrac{1}{2}q^{-1/4}\,\frac{\theta_1(\omega, q)}{\sin \pi\omega},$$

where

$$\theta_3(v, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\pi vi},$$

$$\theta_1(v, q) = \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n q^{(2n+1)^2} e^{(2n+1)\pi vi},$$

and a parallel relation between θ_1 and θ_0.

F. M. Arscott (Battersea, London)

2723:

Chin, Jin H.; Churchill, Stuart W. Incorporation of error function absorption coefficient in transport equation. Quart. Appl. Math. **18** (1960/61), 93–96.

Let σ be a function of λ such that

$$\frac{1}{\Delta\lambda}\int_{\lambda-\Delta\lambda/2}^{\lambda+\Delta\lambda/2} e^{-\sigma x} d\lambda = \operatorname{erfc}\sqrt{(\gamma x)} = \frac{1}{\pi}\int_1^\infty \frac{e^{-\gamma x\eta}}{\eta\sqrt{(\eta-1)}}\, d\eta,$$

where γ depends on λ and $\Delta\lambda$. The authors show, by expansion of F in a series of Laguerre polynomials, that formally

$$\frac{1}{\Delta\lambda}\int_{\lambda-\Delta\lambda/2}^{\lambda+\Delta\lambda/2} F(\sigma x) d\lambda = \frac{1}{\pi}\int_1^\infty \frac{F(\gamma x\eta) d\eta}{\eta\sqrt{(\eta-1)}}$$

for a large class of functions F. This formula is then applied to the problem mentioned in the title.

A. Erdélyi (Pasadena, Calif.)

2724:

Greenhill, Alfred George. ★The applications of elliptic functions. Dover Publications, Inc., New York, 1959. xii+357 pp. Paperbound: \$1.75.

This edition is an unabridged and unaltered republication of the work first published in 1892 [Macmillan, London]. The chapter headings are as follows: (1) The elliptic functions; (2) The elliptic integrals; (3) Geometrical and mechanical illustrations of the elliptic functions; (4) The addition theorem for elliptic functions; (5) The algebraical form of the addition theorem; (6) The elliptic integrals of the second and third kind; (7) The elliptic integrals in general and their applications; (8) The double periodicity of the elliptic functions; (9) The resolution of the elliptic functions into factors and series; (10) The transformation of elliptic functions.

2725:

Stiller, Heinz. Methode zur Ableitung des Additionstheorems der Kugelfunktionen. Z. Angew. Math. Mech. **40** (1960), 277–278.

2726:

Bailey, W. N. On two manuscripts by Bishop Barnes. Quart. J. Math. Oxford Ser. (2) **10** (1959), 236–240.

This paper contains a summary of two lengthy unpublished manuscripts by the late E. W. Barnes. Apparently the work had been done around 1908, but had never been prepared in a form for publication. The first manuscript gave a reduction formula for Appell's hypergeometric function F_4 of two variables as a product of two hypergeometric functions of one variable. Bailey himself had obtained this result independently in 1933. The second manuscript was meant to be a continuation of Barnes' 1908 memoir [Quart. J. Pure Appl. Math. **39** (1908), 97–204]. It contained various formulae expressing the product of two Legendre functions as a contour integral, and related results.

C. A. Swanson (Vancouver, B.C.)

2727:

Jucys, A.; Perkalskis, B.; Šugurovas, V.; Uspalis, K. Calculation of integrals of products of several spherical functions. Vilniaus Valst. Univ. Mokslo Darbai. Mat. Fiz. Chem. Mokslų Ser. **3** (1955), 35–62. (Lithuanian. Russian summary)

The authors show that an integral

$$I_n = \int_0^\pi \int_0^{2\pi} \prod_{i=1}^n Y(k_i, m_i|\theta, \phi) \sin\theta\, d\theta d\phi,$$

occurring in quantum mechanics of an atom, is not zero if and only if

$$\sum_{i=1}^h m_i = 0, \quad k_p - \sum_{\substack{i=1\\ i\neq p,j}}^n k_i \leqq k_j \leqq \sum_{\substack{j=1\\ i\neq j}}^h k_i$$

$$(j, p = 1, 2, \cdots, h, j \neq p).$$

Here

$$Y(k, m|\theta, \phi) = (-1)^m \frac{1}{(2\pi)^{1/2}} e^{im\phi} \frac{((2k+1)(k-m)!)^{1/2}}{2(k+m)!} P_k^m(\cos\theta),$$

where P_k^m is the mth order associated Legendre function (of the Legendre function of order k). The values of I_3 are given by J. A. Gaunt [Phil. Trans. Roy. Soc. A **228** (1929), 151–196; p. 192]. The authors of the present article give the values of I_4 and I_5 for different combinations of m_i's and k_i's. Explicit forms of $Y(k, m|\theta, \phi)$ for $m=1, 2, \cdots, 6$ and $k=\pm 1, \pm 2, \cdots, \pm 6$ are listed.

Č. Masaitis (Aberdeen, Md.)

2728:

Erber, Thomas. Stirling numbers and hypergeometric functions. J. Math. and Phys. **38** (1959/60), 331–334.

The author considers an expansion of the Gauss function $F(a, b; c; z)$ not, as is usual, as a series in terms of z, but as a series in terms of the first parameter a, where the coefficient of a^k is

$$h_k = (-1)^k \sum_{n=k}^{\infty} (b)_n(-z)^n S_n{}^k/[(c)_n n!],$$

and $S_n{}^k$ is a Stirling number. He also gives a differential equation and an integral expressing h_{k+1} in terms of h_k.

L. J. Slater (Cambridge, England)

2729:

Sharma, K. C. Infinite integrals involving E-function and Bessel function. Proc. Nat. Inst. Sci. India. Part A **25** (1959), 337–339.

The object of this paper is the evaluation of the integrals

$$\int_0^\infty e^{-t}t^{k-1}E(\alpha, \beta: :t)E(p; \alpha_r: q; \beta_s: zt^n)dt,$$

$$\int_0^\infty x^{2k-1}K_{2\nu}(ax)E(p; \alpha_r: q; \beta_s: zx^{2n})dx,$$

(in which E is MacRobert's function) in terms of Meijer's G-function. [The results appear to be included in C. S. Meijer's results, Proc. Nederl. Wetensch. **44** (1941), 81–92; MR **2**, 287.]

A. Erdélyi (Pasadena, Calif.)

2730:

Knottnerus, U. J. ★Approximation formulae for generalized hypergeometric functions for large values of the parameters. J. B. Wolters, Groningen, 1960. viii+166 pp. Dutch Guilders 12.50.

This interesting handbook discusses the mathematical theory underlying approximations such as

$${}_2F_1[a, b; c+r; z] = 1+\frac{ab}{r}z+O(r^{-2})$$

for complex values of a, b and z. The first chapter defines the generalized hypergeometric function ${}_pF_q[a_1, \cdots, a_p; b_1, \cdots, b_q; z]$ and discusses some of the multitude of special cases of this function. The second chapter gives approximations, for complex values of the a's, b's and z, of the function

$${}_pF_q[a_1+r, \cdots, a_p+r; b_1+r, \cdots, b_q+r; z],$$

in the cases $q=p-1$, and $q=p$, for large positive values of r. It deals with the Barnes integral, and various other formulae of a similar kind, and in particular it covers the corresponding special cases for the confluent functions ${}_1F_1[r; b; z]$ and ${}_1F_1[-r; b; z]$, together with the Whittaker functions $W_{k\pm r/2, m+r/2}(z)$.

The next two chapters are concerned with Meijer's three great expansion theorems for the general G functions,

$$G_{pq}^{mn}\left(\lambda w\middle| \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\right).$$

These three theorems are stated clearly in detail, but without proofs, and the last part of the book covers some applications of these theorems to give further approximation formulae both for the general ${}_pF_q$ functions and for several special cases.

Though the English is weak in places, the mathematical style is in the best West European tradition, with much detailed analysis and careful statements of the convergence conditions, exceptional cases and other special points of difficulty. The references might have been improved by the addition of some American sources. For example mention might have been made of the work of D. B. Sears, in a similar field. But it is very pleasant to see Meijer's theorems being stated simply and used in a practical way, since they represent the most general theorems yet proved in this function theory, and contain within themselves all the usual expansion theorems of mathematical physics. Because their proofs are long and difficult and not readily accessible, they are as yet almost unknown to most students of the subject. This book is particularly valuable, as it draws attention to these theorems and makes use of them in a comparatively elementary way.

L. J. Slater (Cambridge, England)

2731:

Makai, E. A class of systems of differential equations and its treatment with matrix methods. III. Contiguous systems. Publ. Math. Debrecen **6** (1959), 204–213.

[For part II see same Publ. **5** (1958), 269–274; MR **21** #4268.] The author studies the matrix differential equation $XY'=AY$, where $Y'=dY/dx$, A is a square matrix of constant elements, and $X=xI-\Lambda$, where Λ is a diagonal matrix of constant elements. He points out that many hypergeometric systems can be written in this form.

Two systems, $XY'=AY$ and $XZ'=BZ$, are said to be 'contiguous' if there exists a matrix $M(x)=Px+Q$ (P, Q constant square matrices) so that $Y=M(x)Z$. The main theorem of the paper states that if A possesses a characteristic value different from 0 and 1, and if Λ is not a scalar multiple of the unit matrix, then there exists a (non-trivial) contiguous system to $XY'=AY$. The contiguous system constructed by the author is such that all but one of the characteristic values of B are equal to those of A, and one differs by 1 from the corresponding characteristic value of A.

As an application, the author discusses relations between contiguous hypergeometric functions.

A. Erdélyi (Pasadena, Calif.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 2778, 2861, B3047, B3146, B3147.

2732:

Kozlov, É. M. Substantiation of the method of successive reduction of the order of systems of ordinary differential equations. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 1295–1299. (Ukrainian. Russian and English summaries)

Author's summary: "In the present paper which is the logical conclusion of an earlier paper [same Dopovidi **1958**, 813–816; MR **20** #5930], the author outlines a substantiation of the method of successive reduction of the order. It is shown that a solution obtained by the method of successive reduction of the order may be made more precise by the algorithm of successive approximations, which in the case involved rapidly leads to an exact solution of the system and is relatively simple in practice."

2733:

Opial, Z. Sur un problème de J. Szarski et T. Ważewski. Colloq. Math. 7 (1959/60), 275–276.

The author gives an affirmative answer to the following question of J. Szarski and T. Ważewski [Ann. Soc. Polon. Math. **20** (1947), 157–160; MR **10**, 121]. Consider the system of differential equations $y_i' = f_i(x, y_1, y_2, \cdots, y_n)$ $(i = 1, 2, \cdots, n)$, where each f_i is defined and continuous in an open domain Ω of euclidean $n+1$ space. Assume that through each point of Ω there is a solution of the system which is a line segment. Is it possible for the system to have other solutions than straight lines?

W. R. Utz (Columbia, Mo.)

2734:

Kurcveil', Yaroslav [Kurzweil, Jaroslav]; Vorel, Zdeněk. Continuous dependence of solutions of differential equations on a parameter. Czechoslovak Math. J. 7 (82) (1957), 568–583. (Russian. English summary)

First, the authors generalize the theorem of M. A. Krasnosel'skiĭ and S. G. Kreĭn [Uspehi Mat. Nauk (N.S.) **10** (1955), no. 3 (65), 147–152; MR **17**, 152] on continuous variation of the solution of an initial-value problem for a system of first-order ordinary differential equations with respect to integrally continuous variation of the system's right-hand member. The requirements imposed on the right-hand side are analogous to Carathéodory conditions; the generalization consists essentially in that the initial conditions are varied as well as the right-hand side, and that the domain of the unknown function is not assumed bounded. Next analogous theorems are proved for systems, written in vector form $d^n x/dt^n = X(t, x, \lambda)$; here it is assumed that for arbitrary $\lambda \in \Lambda$ and $x \in D$ there exist vectors $\alpha_0(\lambda), \cdots, \alpha_{n-1}(\lambda)$ for which uniformly in $x \in D$ and $t \in [0, T]$

$$\lim_{\lambda \to \lambda_0} \left\{ \int_0^t \frac{(t-\sigma)^{n-1}}{(n-1)!} X(\sigma, x, \lambda) d\sigma - \sum_{i=0}^{n-1} \frac{\alpha_i(\lambda)}{i!} t^i \right\} = \int_0^t \frac{(t-\sigma)^{n-1}}{(n-1)!} X(\sigma, x, \lambda_0) d\sigma$$

(the other restrictions are natural, and analogous to those made in the previous theorem). It is asserted that, for sufficiently small $\lambda - \lambda_0$ and $x^{(i)}(0, \lambda) - x^{(i)}(0, \lambda_0) + \alpha_i(\lambda)$ $(i = 0, \cdots, n-1)$, there exists in any interval $[0, T]$ a solution $x(t, \lambda)$ for which $x(t, \lambda) - x(t, \lambda_0)$ is uniformly as small as desired. An example is given.

A. D. Myškis (RŽMat **1959** #316)

2735:

Kurzweil, Jaroslav. Generalized ordinary differential equations and continuous dependence on a parameter. Czechoslovak Math. J. 7 (82) (1957), 418–449. (Russian summary)

A generalized Perron integral is introduced. Let $U(\tau, t)$ be defined for $\tau_1 \leqq \tau \leqq \tau_2$, $|t - \tau| \leqq \delta(\tau)$, where $\delta(\tau) > 0$. The function $M(\tau)$ is called an upper function if there exists a function $\delta'(t) > 0$ such that $(\tau - \tau_0)(M(\tau) - M(\tau_0)) \geqq (\tau - \tau_0)(U(\tau, \tau_0) - U(\tau_0, \tau_0))$ for $\tau_1 \leqq \tau_0 \leqq \tau_2$, $|\tau - \tau_0| \leqq \delta'(\tau_0)$. The function $m(\tau)$ is called a lower function if it satisfies the reverse inequality. If upper and lower functions exist and

$$\inf_{M(\tau)} (M(\tau_2) - M(\tau_1)) = \sup_{m(\tau)} (m(\tau_2) - m(\tau_1)) = a,$$

the number a is called the generalized Perron integral of DU and is denoted $\int_{\tau_1}^{\tau_2} DU(\tau, t)$.

The author proves the existence of this integral under certain assumptions on $U(\tau, t)$, and studies its basic properties.

The function $x(t)$ is called a solution of the generalized differential equation (1) $dx/d\tau = DF(x, \tau, t)$ if $x(\tau_2) - x(\tau_1) = \int_{\tau_1}^{\tau_2} DF(x(\tau), \tau, t)$ for arbitrary τ_1 and τ_2. Solution of a system of differential equations is defined analogously. If $f(x, t)$ is continuous and $F(x, \tau, t) = f(x, \tau)t$ (or $F(x, \tau, t) = \int_\tau^t f(x, s) ds$), then any solution of the equation $dx/d\tau = f(x, \tau)$ is a solution of (1) and conversely.

As a special case of (1), the equation (2) $dx/d\tau = DF(x, t)$ is considered. On the assumption that the function $F(x, t) - F(x, t_1)$ satisfies a Lipschitz condition in x, and under some other restrictions, the author proves for (2) an existence theorem and a theorem of continuous dependence of a solution on a parameter; also, for the linear case, a uniqueness theorem.

By this method the author generalizes the theorem of Krasnosel'skiĭ and Kreĭn [Uspehi Mat. Nauk (N.S.) **10** (1955), no. 3 (65), 147–152; MR **17**, 152] on continuous dependence of a solution on a parameter.

A. F. Filipov (RŽMat **1958** #88237)

2736:

Kurcveil', Ya. On generalized ordinary differential equations, possessing discontinuous solutions. Prikl. Mat. Meh. **22** (1958), 27–45. (Russian)

The following definitions are borrowed from the author's paper reviewed next above. Let $\delta(\tau)$ be a positive function on $[\tau_*, \tau^*]$. Let $U(\tau, t)$ be a real function defined on $\tau \in [\tau_*, \tau^*]$, $t \in [\tau - \delta(\tau), \tau + \delta(\tau)]$. The real function $M(\tau)$ (on the same τ range) is an upper function relative to U if there exists a positive function $\delta'(\tau) \leqq \delta(\tau)$, $\tau \in [\tau_*, \tau^*]$ such that, for $\tau_0 - \delta'(\tau_0) \leqq \tau \leqq \tau_0 + \delta'(\tau_0)$, we have

$$(\tau - \tau_0)(M(\tau) - M(\tau_0)) \geqq (\tau - \tau_0)(U(\tau_0, \tau) - U(\tau_0, \tau_0)).$$

The function $m(\tau)$ is a lower function relative to U if $-m$ is an upper function relative to $-U$. Whatever M, m (relative to U) we have:

$$M(\tau^*) - M(\tau_*) \geqq m(\tau^*) - m(\tau_*).$$

If U possesses M, m such that $I = \inf_M (M(\tau^*) - M(\tau_*)) = \sup_m (m(\tau^*) - m(\tau_*))$, then U is said to be Perron integrable in the general sense, and I is said to be the generalized Perron integral of DU between τ_* and τ^*: (1) $I = \int_{\tau_*}^{\tau^*} DU(\tau, t)$. Similarly, in obvious manner, if U is an n-vector. Write down the formal n-vector equation (2) $\dot{x} = DF(x, t)$. By a solution $x(\tau)$ of this equation is meant a function such that $x(\tau_4) = x(\tau_3) + \int_{\tau_3}^{\tau_4} DF(x(\tau), t)$, $\tau_3, \tau_4 \in [\tau_*, \tau^*]$. If $\partial F(x, t)/\partial t = f(x, t)$ is continuous then the solutions of (2) are the same as those of $\dot{x} = f(x, t)$. The author proves the existence of the integral (1), as well as the existence of the solutions of (2) and their continuous dependence upon a parameter, when $F(x, t)$ is bounded for fixed t and fulfills a certain continuity condition with respect to x. In particular, it is shown that the solution of $\dot{x} = f(x, t) + d(t)$, where f is continuous and $d(t)$ is near the Dirac function: $d(t) \geqq 0$, $d(t) = 0$ for $|t| \geqq \delta > 0$, $\int_{-\infty}^{+\infty} d(t) dt = 1$, approximates closely a certain discontinuous function. Under certain conditions, the uniqueness of the solutions of (2) is also established.

S. Lefschetz (Princeton, N.J.)

2737:

Kurzweil, Jaroslav. On integration by parts. Czechoslovak Math. J. 8 (83) (1958), 356–359. (Russian summary)

Let $U(\tau, t)$ be defined for $\tau \in \langle \tau_*, \tau^* \rangle$, $t \in \langle \tau_*, \tau^* \rangle$, let $\int_{\tau_*}^{\tau^*} D_t U(\tau, t)$ denote the generalised Perron integral introduced by the author in his previous paper [#2735 above] and let $V(\tau, t) = U(\tau, \tau) - U(\tau, t) - U(t, \tau) + U(t, t)$. Theorem: Suppose that two of the integrals occurring in

$$(*)\quad \int_{\tau_*}^{\tau^*} D_t U(\tau, t) + \int_{\tau_*}^{\tau^*} D_\tau U(\tau, t) = U(\tau^*, \tau^*) - U(\tau_*, \tau_*) - \int_{\tau_*}^{\tau^*} DV(\tau, t)$$

exist. Then the third one exists and (*) holds. Letting $U = f(\tau)\varphi(t)$, f, φ of bounded variation, the author derives as a corollary the known theorem: $\int_{\tau_*}^{\tau^*} f d\varphi + \int_{\tau_*}^{\tau^*} \varphi df = [f\varphi]_{\tau_*}^{\tau^*}$, where one of the functions is continuous (or both are normalized). *M. Cotlar* (Buenos Aires)

2738:

Kurzweil, Jaroslav. Generalized ordinary differential equations. Czechoslovak Math. J. 8 (83) (1958), 360–388. (Russian summary)

In an earlier paper [#2735 above] the author defined generalized differential equations. In this paper he deals with a special class of generalized differential equations, viz., $dx/dt = DF(x, t)$ where $F(x, t)$ satisfies

$$\|F(x, t_2) - F(x, t_1)\| \leq |h(t_2) - h(t_1)|,$$

$$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| \leq \omega(\|x_2 - x_1\|)|h(t_2) - h(t_1)|$$

with $h(t)$ increasing, continuous from the left, and $\omega(\eta)$ increasing, continuous and $\omega(0) = 0$. He proves theorems of existence and continuous dependence on a parameter. Results on the continuous dependence on a parameter are then used to examine the behavior of solutions of the sequence of classical differential equations $dx/dt = f(x, t) + g(x)\varphi_k(t)$ where $\varphi_k(t)$ tends to the Dirac function. These results are very interesting but unfortunately too long to be stated here. *M. Zlámal* (Brno)

2739:

Doležal, Václav; Kurzweil, Jaroslav; Vorel, Zdeněk. The Dirac function in non-linear differential equations. Apl. Mat. 3 (1958), 348–359. (Czech. Russian and English summaries)

In the real nonlinear system

$$(1)\quad \frac{dx_i}{dt} = f_i(x_1, x_2, \cdots, x_n, t) + g_i(x_1, \cdots, x_n)h(t) \quad (i = 1, \cdots, n),$$

let f_i, g_i be continuous for $-\infty < x < \infty$, $-\infty < t < \infty$ and let $h(t) = \lim_{k\to\infty} h_k(t)$, h_k continuous, and (2) $h_k(t) \geq 0$, $\int_{-t_1}^{t} h_k(\tau)d\tau \to 0$ for $t < 0$, $\int_{-t_1}^{t} h_k(\tau)d\tau \to 1$ for $t > 0$, $t \in (-t_1, t_2)$, $t_1 > 0$, $t_2 > 0$. If h in (1) is replaced by h_k, let $[x_{k1}(t), \cdots, x_{kn}(t)]$ be the unique solution satisfying the initial conditions (3) $x_{ki}(-t_1) = \bar{x}_i$ $(i = 1, \cdots, n)$. Under additional conditions involving existence of solutions of (5) and (6) below, J. Kurzweil has shown [see preceding review] that the solutions $x_{ki}(t)$ approach functions $x_i(t)$ $(i = 1, \cdots, n)$ independent of the form of the functions $h_k(t)$, $-t_1 \leq t \leq t_2$, $t \neq 0$. Using the special sequence (4) $h_k(t) = k$, $|t| \leq 1/2k$, and $h_k(t) = 0$, $|t| > 1/2k$ $(k = 1, 2, \cdots)$, the authors show that the above limit functions $x_i(t)$ $(i = 1, \cdots, n)$ are for $t < 0$ solutions of (5) $du_i/dt = f_i(u_1, \cdots, u_n, t)$ $(i = 1, \cdots, n)$ satisfying (3). At $t = 0$ each $x_i(t)$ has a jump given by $v_i(\frac{1}{2}) - v_i(-\frac{1}{2})$, where $v_i(t)$ are solutions of the autonomous system (6) $dx_i/d\tau = g_i(x_1, \cdots, x_n)$ $(i = 1, \cdots, n;\ -\frac{1}{2} \leq \tau \leq \frac{1}{2})$ with initial conditions $v_i(-\frac{1}{2}) = \lim_{t\to 0-} x_i(t)$. For $t > 0$ the $x_i(t)$ are solutions of (5) and $\lim_{t\to 0+} x_i(t) = v_i(\frac{1}{2})$. Interesting applications to problems in circuit theory, aerodynamics, and atomic physics are considered in detail. *J. A. Nohel* (Atlanta, Ga.)

2740:

Kurzweil, Jaroslav. Unicity of solutions of generalized differential equations. Czechoslovak Math. J. 8 (83) (1958), 502–509. (Russian summary)

The author has introduced the concept of generalized differential equations in an earlier paper [#2735 above]. The present paper contains two uniqueness theorems concerning regular solutions $x(t)$ of the initial value problem consisting of the generalized differential equation $dx/dt = DF(x, t)$ on $t_1 \leq t \leq t_2$ and the initial condition $x(t_0) = c$. The second theorem gives a criterion for the uniqueness in the case of classical differential equations also. *J. B. Diaz* (College Park, Md.)

2741:

Vrkoč, Ivo. A note to the unicity of generalized differential equations. Czechoslovak Math. J. 8 (83) (1958), 510–512. (Russian summary)

J. Kurzweil [see preceding review] proved two uniqueness theorems for generalized differential equations. The author presents an example to show that if the assumption $x(t_0) = c$ is omitted then theorem I of that paper is not true. *J. B. Diaz* (College Park, Md.)

2742:

Kurzweil, Jaroslav. Addition to my paper "Generalized ordinary differential equations and continuous dependence on a parameter". Czechoslovak Math. J. 9 (84) (1959), 564–573. (Russian summary)

The present paper contains an improved treatment of the results of section 3 of the paper mentioned in the title [#2735 above], which are proved here under weaker monotonicity assumptions. This implies that the results of section 4 of the previous paper are also valid in more general form. The derivation of the uniqueness and of the variation-of-constants formula for solutions of generalized linear differential equations given in section 5 of the previous paper are stated to be incorrect, and new versions of these proofs are presented. *J. B. Diaz* (College Park, Md.)

2743:

Chaléat, Raymond. Seconde approximation de la perturbation d'amplitude d'un oscillateur quasi linéaire. C. R. Acad. Sci. Paris 250 (1960), 1177–1179.

An expression is given for the second approximation to the perturbations of the amplitude and phase of the quasi-linear equation $\theta'' + \omega^2\theta = \lambda\omega^2 f(\theta, \theta')$. This is applied to van der Pol's equation. *J. P. LaSalle* (Baltimore, Md.)

2744:

Coppel, W. A. The operational solution of linear differential equations with constant coefficients. Ann. Polon. Math. 7 (1959), 113–126.

The author gives an exposition of Heaviside's operational calculus as far as it is needed for the solution of ordinary linear differential equations with constant coefficients. His presentation is based on a definition and properties of rational functions of the operator of differentiation, D. {Although the author mentions Boole as having used rational functions of D, he does not refer to presentations, somewhat similar to his own, of "symbolic calculus" which are found in several elementary textbooks on differential equations; see, for instance, R. P. Agnew, *Differential equations* [McGraw-Hill, New York, 1960], Chapter 6.} *A. Erdélyi* (Pasadena, Calif.)

2745:

Goertzel, Gerald; Tralli, Nunzio. ★Some mathematical methods of physics. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1960. xiii + 300 pp. \$8.50.

This book is an outcome of a graduate course of lectures given by one of the authors at New York University and is concerned with the basic concepts involved in the study of linear differential equations, emphasis being placed on eigenvalues, eigenfunctions and Green's functions. The book is in three sections. The first studies systems with a finite number of degrees of freedom with the use of matrix and operator concepts. The authors are concerned with the solution of systems of linear differential equations with constant coefficients having a finite number of dependent variables and one independent variable. Such a set of equations is written in matrix notation and its solution developed by means of the concept of a function of a matrix, the evaluation of such a function being discussed. The remaining chapters of this section give an introduction to vector spaces and linear operators.

Part II considers systems with an infinite number of degrees of freedom and is thus concerned with methods of solving linear partial differential equations. By approximating to a continuous system by one containing discrete units the authors obtain the equations of motion of the former system by use of results in part I and are thus able to extend the concepts developed for discrete systems to continuous systems besides introducing new concepts. The authors first discuss operators in continuous systems and then consider the Laplacian operator in various domains in one, two and three dimensions. This leads to some discussion of Fourier analysis, cylindrical functions and spherical harmonics. The remaining chapters deal with Green's functions and radiation and scattering problems.

In Part III the authors consider various approximation methods (perturbation of eigenvalues, variational methods, iteration and numerical procedures) which can be used to estimate the eigenvalues and eigenfunctions which arise in the problems of the first two parts and in other problems. Each chapter of the book is followed by a set of examples and a list of references to further reading and certain background and supplementary material is included in the form of appendices.

The book is stated by the authors to be intended for physicists concerned with such fields as quantum mechanics, acoustics, electromagnetic theory, reactor physics, etc., and provides a good introduction to the techniques used in these branches of physics.

W. D. Collins (Newcastle-upon-Tyne)

2746:

Poole, E. G. C. ★Introduction to the theory of linear differential equations. Dover Publications, Inc., New York, 1960. viii + 202 pp. \$1.65.

Unaltered and unabridged republication of the first edition [Oxford Univ. Press, London, 1936].

2747:

Kurzweil, J. Linear differential equations with distributions as coefficients. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 7 (1959), 557–560. (Russian summary, unbound insert)

This is a brief summary of results which will be presented in detail in the Czechoslovak Mathematical Journal. It seems appropriate to give here a rough idea of the nature of the work and defer a more accurate account until the appearance of the complete paper.

The author considers a linear differential system

$$dx_i/dt = \sum_{j=1}^{k} a_{ij}x_j \quad (i = 1, 2, \cdots, k),$$

on a compact interval. The coefficients belong to certain Banach spaces of distributions. These are, roughly speaking, L^p functions and (distribution) derivatives and integrals of arbitrary order of L^p functions. The author proves that all solutions of the system also belong to such Banach spaces of distributions, establishes, under appropriate conditions, existence and uniqueness theorems, shows that if the integrals of the coefficients possess values in the sense of Łojasiewicz then the solutions also possess such values, and formulates an existence and uniqueness theorem for a differential equation of order n.

A. Erdélyi (Pasadena, Calif.)

2748:

Greguš, Michal. Oszillatorische Eigenschaften der Lösungen der linearen Differentialgleichung dritter Ordnung $y''' + 2Ay' + (A' + b)y = 0$, wo $A = A(x) \leq 0$ ist. Czechoslovak Math. J. 9 (84) (1959), 416–428. (Russian summary)

The author examines the oscillatory behavior of the solutions of the equation

$$(1) \qquad y''' + 2A(x)y' + [A'(x) + b(x)]y = 0,$$

where $A(x) \leq 0$, $A'(x)$, $b(x) \geq 0$ are continuous functions of $x \in (-\infty, \infty)$. One of the main results is that if $b - A' \leq 0$ then every solution of (1) has at most two simple zeros or one double zero in the interval $(-\infty, \infty)$. Further the author finds a sufficient condition which guarantees that every solution $y(x)$ of (1) with the property $y(a) = 0$, $a \in (-\infty, \infty)$ has infinitely many zeros for $x > a$. In the third part of the paper there are proved a comparison theorem for two equations of the form (1) and an oscillation theorem for an equation of the form (1) with coefficients depending on a parameter λ. By means of the oscillation theorem, the author solves a three-point boundary problem for an equation (1).

M. Zlámal (Brno)

2749:

Biot, M. A.; Tolstoy, I. Canonical and Hamiltonian formalism applied to the Sturm-Liouville equation. Quart. Appl. Math. **18** (1960/61), 163–172.

A Sturm-Liouville equation is equivalent to a Riccati equation, and the essence of the authors' procedure is, in effect, to consider transformations of the latter, suitable for heuristic approximation in various circumstances.

T. M. Cherry (Melbourne)

2750:

Opial, Z. Sur un problème aux limites pour l'équation differentielle du second ordre. Ann. Polon. Math. **7** (1960), 223–231.

Consider the boundary-value problem (1) $u''=f(t, u, u')$, $u'(a)=A$, $u(b)+hu'(b)=B$. Let the function $f(t,u,v)$ be continuous in the region $a\leq t\leq b$, $-\infty<u<\infty$, $-\infty<v<\infty$, and have continuous derivatives f_u, f_v which are supposed to satisfy there $|f_u(t, u, v)|\leq M$ and $f_v(t, u, v)\leq K$ $(K\geq 0)$. Under these assumptions the author proves that the problem (1) has exactly one solution if the length of $\langle a, b\rangle$ is smaller than that of the interval in which the solution $w(t)$ of $w'=w^2+Kw+M$, $w(0)=0$, satisfies the inequality $w(t)\leq 1/h$, or is smaller than the length of the interval of existence of the solution $w(t)$, according as $h\neq 0$ or $h=0$. This theorem represents a generalization of a result of C. Corduneanu [An. Şti. Univ. "Al. I. Cuza" Iaşi, Secţ. I. (N.S.) **1** (1955), 11–16; MR **18**, 737]. Further results concern the asymptotic behavior of the solutions of $u'=f(t, u)$.

M. Zlámal (Brno)

2751:

Buravs'kiĭ, É. S. On the construction of bilateral approximations for eigenvalues of the one-dimensional boundary problem. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 1047–1050. (Ukrainian. Russian and English summaries)

Author's summary: "The author considers the problem of eigenvalues connected with a pair of differential expressions. Upper and lower estimations are proposed for an approximate calculation of the eigenvalues, and the existence of more precise estimates for the eigenvalues of the problem under consideration is indicated."

2752a:

Chin, Yuan-shun. On differential equations defined on the torus. Sci. Sinica **8** (1959), 661–689.

A translation into English of Acta Math. Sinica 8 (1958), 348–368 [MR **21** #3622].

L. W. Green (Minneapolis, Minn.)

2752b:

Chin, Yuan-Shun. Sur les équations différentielles à la surface du tore. I. Sci. Record (N.S.) **1** (1957), no. 3, 7–11.

French summary of the above.

2753:

Klotter, K.; Kreyszig, E. Amplitudes of oscillations governed by a modified van der Pol equation. Quart. Appl. Math. **18** (1960/61), 61–69.

It is known that the equation

$$\ddot{q}-k^2s \operatorname{sgn} \dot{q}(1-\beta^2\dot{q}^2)+k^2f(q) = 0$$

has a unique limit cycle under the assumption that $f(q)$ is a "genuine" restoring force $(qf(q)\geq 0,\ f(q)\neq 0)$. The equation has a first integral which gives a relation between two consecutive maximum displacements (consecutive amplitudes). This is discussed when $f(q)=\mu^{n-1}q^n$ (n odd) and when $f(q)$ is a sum of two such terms.

J. P. LaSalle (Baltimore, Md.)

2754a:

Barbălat, I. Systèmes d'équations différentielles des oscillations non linéaires. Com. Acad. R. P. Romîne **9** (1959), 779–782. (Romanian. Russian and French summaries)

2754b:

Barbălat, I. Systèmes d'équations différentielles d'oscillations non linéaires. Rev. Math. Pures Appl. **4** (1959), 267–270.

[Romanian and French versions of the same article.] Consider the equation $x''+H(x)x'+Ax=e(t)$, wherein A is a real, positive definite, $n\times n$ matrix, $H(x)$ is the hessian matrix of a real, class C^2 function $u(x)$ defined for all $x\in R^n$, and $e(t)$ is an $n\times 1$ continuous matrix defined for all $t\geq 0$. Suppose $AH=HA$ for all $x\in R^n$. If $H(x)$ is positive semi-definite for all $x\in R^n$ and if the equation has a solution $\varphi(t)$, defined for all $t\geq 0$, for which $\|\varphi(t)\|$ is bounded, then for all solutions of the equation $x(t)$, defined for all $t\geq 0$, $\|x(t)\|$ is bounded. Moreover, if $\|\varphi'(t)\|$ is bounded, $\|x'(t)\|$ is bounded for $t\geq 0$. If the matrix $H(x)$ is positive definite except possibly at isolated points of R^n and if the equation has a solution $\varphi(t)$, defined for all $t\geq 0$, for which both $\|\varphi(t)\|$ and $\|\varphi'(t)\|$ are bounded, then for each solution $x(t)$ of the equation defined for all $t\geq 0$,

$$\lim_{t\to\infty} \|x(t)-\varphi(t)\| = \lim_{t\to\infty} \|x'(t)-\varphi'(t)\| = 0.$$

These theorems generalize theorems of R. Reissig [Math. Nachr. **13** (1955), 313–318; MR **17**, 38] and the reviewer [Ann. Mat. Pura Appl. (4) **42** (1956), 313–324; MR **18**, 576].

W. R. Utz (Columbia, Mo.)

2755:

Moore, Richard A.; Nehari, Zeev. Nonoscillation theorems for a class of nonlinear differential equations. Trans. Amer. Math. Soc. **93** (1959), 30–52.

This investigation of the solutions of (*) $y''+p(x)y^{2n+1}=0$, where $p(x)>0$ is continuous in $(0,\infty)$ and n is a positive integer, falls into two parts. The first group of results depends on methods similar to those used by the reviewer [Pacific J. Math. **5** (1955), 643–647; MR **17**, 264] in connection with the criteria $\int^\infty xp(x)dx<\infty$, $\int^\infty x^{2n+1}p(x)dx<\infty$ for at least one or for all solutions to be non-oscillatory; here further conclusions are drawn, and the reviewer's requirement (in the second case) that $p(x)$ be differentiable and monotonic is dropped. It is shown that $\int^\infty x^{2n+1}p(x)dx<\infty$ is equivalent to the existence of solutions such that $x^{-1}y(x)\to\text{const}>0$ as $x\to\infty$, and that this criterion ensures the existence of "properly non-oscillatory" solutions. These have at any rate one zero, but are ultimately of one sign; the category of solutions with constant sign is viewed as too wide.

Another result in this section is that if there is a non-oscillatory solution for which $\int^\infty x^\nu \{y(x)\}^{-2n} dx < \infty$, for some $\nu > -1$, then $\int^\infty x^{\nu+2} p(x) dx < \infty$.

The latter half of the paper exploits the approach of minimising the "generalised Rayleigh quotient" $J(u) = \{\int_a^b u'^2 dx\}^{n+1} / \int_a^b u^{2n+2} p(x) dx$ [cf. Z. Nehari, Trans. Amer. Math. Soc. **85** (1957), 428–445; MR **19**, 415]. The minimum subject to $u(a)=0$ is achieved by a solution of (*) such that $y(a)=y'(b)=0$, with an analogous result for the case $u(a)=u(b)=0$; in both cases the solution is to be positive in (a, b) but need not be unique. Making $b\to\infty$, a connection appears between the eventualities, firstly that $J(y)$ has a positive lower bound for all $b>a$, and secondly that (*) has a solution vanishing at $x=a$ and positive for $x>a$. This matter is partly clarified for the case that $\int^\infty x^{n+1} p(x) dx < \infty$. Finally, precise bounds are found for $J(y)$ in cases in which (*) admits explicit integration.

F. V. Atkinson (Toronto)

2756:

Nehari, Zeev. On a class of nonlinear second-order differential equations. Trans. Amer. Math. Soc. **95** (1960), 101–123.

This treatment of $y''+yF(y^2, x)=0$ is closely parallel to the paper of Moore and Nehari [see the preceding review] on the special case $F(t, x)=t^n p(x)$; however for the variational approach a distinct functional is used. $F(t, x)$ is assumed continuous for $t\geq 0$, $x>0$ and positive for $t>0$, $x>0$, and such that $t^{-\varepsilon}F(t, x)$ is increasing for fixed $x>0$ and some $\varepsilon>0$. The existence of bounded non-oscillatory solutions is shown to be equivalent to $\int^\infty xF(c, x)dx < \infty$, for some positive c. In generalization of the criterion $\int^\infty x^{2n+1}p(x)dx<\infty$, one has $\int^\infty tF(ct^2, t)dt < \infty$ either for some $c>0$, or again for all $c>0$; the former ensures the existence of properly non-oscillatory solutions, the latter that they are all either bounded or of the asymptotic form βx, and that if in addition $F(\eta, x)$ is decreasing in x, then all solutions are non-oscillatory.

For the variational aspect, the author considers $\lambda(a, b) = \min \int_a^b [y'^2 - G(y^2, x)]dx$, where $G(\eta, x) = \int_0^\eta F(t, x)dt$, and y is subject to $y(a)=0$, $y(x) \in D^1[a, b]$, and is normalised by $\int_a^b y'^2 dx = \int_a^b y^2 F(y^2, x) dx$. The minimum is achieved by a solution for which $y(a) = y'(b) = 0$, $y(x)>0$ in $(a, b]$. Since $\lambda(a, b)$ is decreasing in b, one may define $\lambda(a) = \lim \lambda(a, b)$ as $b\to\infty$. Noteworthy results are that if there is a solution vanishing for $x=a$ and positive thereafter, then $\lambda(a)>0$, and that this in turn is equivalent to $x\int_x^\infty F(mt, t)dt \leq M$ for some m and M. On the other hand, this does not ensure the existence of properly non-oscillatory solutions. For the latter it is sufficient that $\int^\infty xF(\beta x, x)dx<\infty$ for all positive β; with this condition there is for every a a bounded solution whose last zero is at $x=a$. The work closes with two results concerning solutions $y(x)$ for which $\lim \inf_{x\to\infty} x^{-1/2}y$ has a positive upper or lower bound.

F. V. Atkinson (Toronto)

2757:

Sideriades, Lefteri. Méthodes topologiques et applications. Ann. Télécommun. **14** (1959), 185–207.

A resumé is given of some of the things known about the nature of the singular points of systems of differential equations of dimension 2 and 3. Particular examples arising from electronics and hydraulics are discussed.

J. P. LaSalle (Baltimore, Md.)

2758:

Blair, K. W.; Loud, W. S. Periodic solutions of $x''+cx'+g(x)=Ef(t)$ under variation of certain parameters. J. Soc. Indust. Appl. Math. 8 (1960), 74–101.

It is known that the system $x''+g(x)=E\cos t$ possesses three periodic solutions of period 2π for sufficiently small positive E and $g(x)$ satisfying certain conditions. The authors first consider the movement of these three periodic solutions as E is increased. They show that there exists an $E_0>0$ such that as E tends to E_0, two of the periodic solutions coalesce and then disappear. The third remains for all positive E and moves in a certain manner.

The periodic solutions described above may be considered as fixed points of an appropriate transformation. Thus the authors consider a mapping of the (x, y) plane into itself which sends a region bounded by a simple closed curve Γ interior to itself. The map is assumed to possess three fixed points interior to Γ, two of which are completely stable. The boundaries of the domains of attraction of these two fixed points are then studied.

Finally an approximate method for locating the periodic solutions of the system $x''+cx'+g(x)=E\cos t$ is given. This method is known to be accurate for linear and almost linear systems. Several observations along with examples worked out on a computer suggest that the procedure may also serve to predict the location of periodic solutions in the case where $g(x)$ is nonlinear.

J. C. Lillo (Lafayette, Ind.)

2759:

Mitropol's'kiĭ, Yu. O.; Likova, O. B. On periodic solutions of nonlinear systems close to the autonomic. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 1175–1178. (Ukrainian. Russian and English summaries)

Authors' summary: "The authors consider a system of nonlinear differential equations. The existence, uniqueness and stability of the periodic solution for this system is proved; and an estimation is given of the difference between this solution and its first approximation."

2760:

Čžan, Sy-in. On estimates of the solutions of systems of differential equations of the accumulation of disturbances and the stability of motion over a finite time interval. Prikl. Mat. Meh. **23** (1959), 640–649 (Russian); translated as J. Appl. Math. Mech. **23**, 920–933.

It is assumed for the n-dimensional linear system $\dot{x}=P(t)x$ that $P(t)$ has simple characteristic roots. Transforming to canonical form and using a Liapunov function $V=e^{-\alpha(t)}(y_1^2+\cdots+y_n^2)$, the author derives a sufficient condition for stability over a finite time (given a_i, $|x_i(t_0)|<a_i$ implies $|x_i(t)|<a_i$ for all t in $[t_0, T]$ and $i=1, \cdots, n$). To apply the condition one must compute a "smallest" characteristic root and the equations of transformation to canonical form. Stability conditions are obtained also for systems with slowly varying coefficients $(P(t)=C+\varepsilon B(t))$ and for systems with a continuously acting disturbance. The application to nonlinear systems is discussed. The examples given are: (1) a slowly varying system of second order; and (2) a second order linear system with constant coefficients and a continuously acting disturbance.

J. P. LaSalle (Baltimore, Md.)

2761:

Reissig, Rolf. Ein Kriterium für asymptotische Stabilität. Z. Angew. Math. Mech. **40** (1960), 94–99.

In some stability problems it appears to be simpler to piece together a Liapunov function without requiring it to be continuous. In this paper a criterion is given for asymptotic stability of an equilibrium position of a nonautonomous system. The Liapunov function is required to satisfy the conditions piecewise by regions. Because of the conditions of jump from one region to the other the method is not a direct method. Two of the conditions involve line integrals along solutions. However, as is illustrated by an example, it may be possible to verify the conditions without solving the differential equations themselves, but it may be necessary to solve differential inequalities. {There is an error in the statement of condition 3. It should be "$-V'$ stellt eine positiv-semidefinite Funktion dar."}

J. P. LaSalle (Baltimore, Md.)

2762:

Opial, Z. Sur la stabilité des solutions périodiques et presque-périodiques de l'équation différentielle $x''+F(x')+g(x)=p(t)$. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 495–500. (Russian summary, unbound insert)

The author uses some results of G. E. H. Reuter [J. London Math. Soc. **27** (1952), 48–58; MR **13**, 844] to study the equation $x''+F(x')+g(x)=p(t)$. Under relatively strong assumptions, too lengthy to list here, he establishes the existence, uniqueness and asymptotic stability of a periodic solution or almost periodic solution when $p(t)$ is periodic or almost periodic, respectively. Besides those conditions which ensure the ultimate boundedness of all solutions, one involves $F'(y)$, $g'(x)$ and $g''(x)$.

C. E. Langenhop (Princeton, N.J.)

2763:

Barbašin, E. A.; Tabueva, V. A. The conditions for the existence of limit cycles. Prikl. Mat. Meh. **23** (1959), 826–835 (Russian); translated as J. Appl. Math. Mech. **23**, 1185–1197.

The pendulum equation $\ddot\theta+a\dot\theta-L+b\sin\theta=0$ has been studied by Hayes, Seifert, Urabe, and others. In most of these cases the equation was studied in the $(\theta,\dot\theta)$ phase plane. By appropriate translation one has on the θ axis in the $(\theta,\dot\theta)$ plane an interval of length 2π which contains the origin and contains three singular points. The origin is a singular point of index $+1$, and the endpoints of the interval are saddle points. It is known that the qualitative behavior of the separatrices leaving these saddle points determines the qualitative behavior of the solutions in the phase plane.

The authors consider a generalization of the equation, $\ddot x+a\dot x+f(x)=0$, where $f(x)=b[\sin(x+\theta_0)-\sin\theta_0]$ for $\dot x>0$ and $f(x)=b[\sin(x+\theta_0)-\sin\theta_0]+2M$ for $\dot x\leqq 0$. There then arise on the θ axis discontinuities in the vector field due to the definition of $f(x)$. Thus, the three singular points in an interval of length 2π about the origin for the pendulum equation are replaced by three singular intervals I_1, I_2, and I_3. By appropriate choice of coordinates the intervals I_1 and I_3 contain the endpoints of an interval of length 2π. I_2 is contained in the interior of this interval. The authors show that there are distinguished solutions which tend to the endpoints of the intervals I_1 and I_3. These solution curves are called separatrices. The paper shows that the qualitative behavior of the solutions in the phase plane is determined by the qualitative behavior of these separatrices. In particular it determines the existence or nonexistence of limit cycles. *J. C. Lillo* (Lafayette, Ind.)

2764:

Urabe, Minoru. On the nonlinear autonomous system admitting of a family of periodic solutions near its certain periodic solution. J. Sci. Hiroshima Univ. Ser. A **22**, 153–173 (1958).

The author continues his previous work on the subject [Funkcial. Ekvac. **1** (1958), 1–84; MR **20** #5921]. By means of suitable orthonormal reference systems moving along a periodic solution of an autonomous system and certain periodic changes of variables, the equations of the neighboring solutions are reduced to simple forms which are applied to the discussion of perturbed systems (existence and stability of periodic solutions). The two cases where the system is fully (i.e., all the neighboring solutions are periodic) or partially (i.e., they form a manifold of dimension less than that of the space) oscillatory are considered in detail. The results are very difficult to summarize. *J. L. Massera* (Montevideo)

2765:

Šimanov, S. N. A generalization of a proposition by Lyapunov on the existence of periodic solutions. Prikl. Mat. Meh. **23** (1959), 409–411 (Russian); translated as J. Appl. Math. Mech. **23**, 583–587.

Let (1) $\dot x=Ax+X(x)$ be a system, where x is an n-vector, A a constant $n\times n$-matrix, and X a function whose expansion begins with terms of at least second degree. Assume that A has m critical roots of the form $\pm N\lambda i$, where N are non-negative integers which include $N=1$, and λ is real; that the corresponding elementary divisors are linear; and that the other roots have negative real parts. Assume that (1) has $m-1$ first integrals $M_k(x)+\Phi_k(x)=C_k$, where M_k are independent first integrals of the linear approximation, corresponding to the roots $\pm N\lambda i$, represented by forms of the first or second degree, and Φ_k are power series starting with terms of degree greater than that of M_k. Then (1) has a family of periodic solutions depending on $m-1$ real parameters, the period being a holomorphic function of these parameters; $x=0$ is stable, and any solution starting from some neighborhood tends as $t\to+\infty$ to a periodic solution of the family. The case where only $<m-1$ first integrals are given is also considered.

The translation is very unsatisfactory.

J. L. Massera (Montevideo)

2766:

Vejvoda, Otto. On the existence and stability of the periodic solution of the second kind of a certain mechanical system. Czechoslovak Math. J. **9** (**84**) (1959), 390–415.

The system which is dealt with has the form

$$(1)\qquad \begin{aligned}\ddot x+x&=\varepsilon f(x,\dot x,\varphi,\dot\varphi,\varepsilon),\\ \ddot\varphi&=\varepsilon M(\dot\varphi)+\varepsilon^2 g(x,\dot x,\varphi,\dot\varphi,\varepsilon),\end{aligned}$$

f and g being 2π-periodic in φ. It represents a generalization of a system which describes the motion of a motor-driven mechanical system assuming that the influence of

the motion of this system on the motor action cannot be neglected. The author seeks the solution of (1) in the form

$$(2) \qquad x = x(t, \varepsilon), \quad \varphi = \omega(\varepsilon)t + \varepsilon\Phi(t, \varepsilon).$$

Under the assumption of the smoothness of the functions f, g and M and under the assumption that the equation $M(\omega_0)=0$ has at least one real positive root $\omega_0^* \neq N/n$ (N, n being natural numbers) which is simple, he proves that there exists only one solution of (1) having the form (2) for which the functions x and Φ are periodic in t of period $T(\varepsilon)=2\pi N/\omega(\varepsilon)$ and are continuous in t and ε for any $t \geqq 0$ and sufficiently small ε, for which $\omega(\varepsilon)$ is continuous for sufficiently small ε and $x(t, 0)=0$, and for which $\omega(0)=\omega_0^*$ holds. Moreover, if the functions f, g, M are analytic in their arguments the solution is analytic in ε for all $t \geqq 0$ and for sufficiently small ε. The author also gives full conditions sufficient for asymptotic orbital stability of the solution. He examines the stability by means of a modified Andronov-Vitt theorem.

M. Zlámal (Brno)

2767:

Skačkov, B. N. On the region of stability of some nonlinear systems of regulation. Vestnik Leningrad. Univ. **15** (1960), no. 1, 100–103. (Russian. English summary)

The system under consideration is (1) $\dot\eta = r\eta + n\xi$, $\dot\xi = f(\eta, \xi)$, $\sigma = p\eta - \xi$, where f is an arbitrary continuous function such that $f(\eta, \xi)]_{\sigma=0} = 0$; $\sigma f(\eta, \xi)]_{\sigma\neq0} > 0$ and r, n, p are real constants such that $r<0$, $n>0$. The author [same Vestnik **13** (1957), no. 3, 67–80; MR **19**, 417] has proved that the system is stable in the large if (2) $1+|np|/r>0$. In the present paper he shows that if the parameters are such that $>$ in (2) is replaced by $<$ there always exists a function $f(\eta, \xi)$ making (1) unstable. [Additional reference: Erugin, Prikl. Mat. Meh. **16** (1952), 620–628; MR **14**, 376.]

S. Lefschetz (Princeton, N.J.)

2768:

Van, Mu-Cyu. The problem of factorization of a system in the theory of stability. Sci. Record (N.S.) **4** (1960), 1–7. (Russian)

This is a comparison of the solutions of (1) $\dot x = Ax$, $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$, where all the matrices are constant, with those of (2) $\dot y = By$, $\dot z = Ez$. The general idea is to replace the stability treatment of (1), when the order of A is high, by the same for (2). By lowering the order one may (presumably) determine more easily all the essential signs of the real parts of the characteristic roots. A couple of theorems are given showing that under certain circumstances the substitution of (2) for (1), in relation to stability, is feasible—and of course practical.

S. Lefschetz (Princeton, N.J.)

2769:

Masani, P. On a result of G. D. Birkhoff on linear differential systems. Proc. Amer. Math. Soc. **10** (1959), 696–698; addendum, 999.

The result stated by Birkhoff is that every system $Y' = P(z)Y$ with a singular point of rank $q+1$ at $z=\infty$ $(q \geqq -1)$ is equivalent at $z=\infty$ to a canonical system $Z' = Q(z)Z$, where $zQ(z)$ is a polynomial of degree less than or equal to $q+1$. The author gives an example to show that this is incorrect in case $q=-1$, and then remarks that the result is valid provided a further restriction is placed on the system. In the addendum the author notes that in 1953 F. R. Gantmaher pointed out this error in his book *Teoriya matric* [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953; MR **16**, 438], a fact the author was unaware of when he made his discovery.

E. A. Coddington (Los Angeles, Calif.)

2770:

Zadiraka, K. V. On the existence and uniqueness of a periodic integral manifold of a system of differential equations with a small parameter in the derivatives. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 1179–1183. (Ukrainian. Russian and English summaries)

Author's summary: "The author considers the system of equations

$$(1) \qquad \frac{dx}{dt} = f(t, x, z), \quad \mu\frac{dz}{dt} = F(t, x, z),$$

and the corresponding degenerate system

$$(2) \qquad \frac{d\bar x}{dt} = f(t, \bar x, \bar z), \quad \bar z = \varphi(t, \bar x),$$

where x and f are n-dimensional, while z and F are m-dimensional vectors, μ is a small parameter, $z=\varphi(t, x)$ is the root of the system $F(t, x, z)=0$. It is assumed that in the region $-\infty<t<\infty$, $x\in G$, $|z-\varphi(t, x)| \leqq \rho$, $0<\mu<\mu^*$, the vectors f, F and φ are sufficiently smooth, the roots of the characteristic equation $\det \|pE - \mathfrak{A}\| = 0$, where the matrix $\mathfrak{A}(t, x) = F_z'(t, x, \varphi+\xi)|_{\xi=0}$ and E, the unique matrix, have negative real parts, the vectors f, F and φ are periodic in t with a period of 2π.

"The author proves: (1) the existence and uniqueness of a periodic integral manifold $z(t, x, \mu) = \varphi(t, x) + \psi(t, x, \mu)$ of system (1) tending toward the integral manifold of system (2) as $\mu\to0$; (2) the existence of limited and uniformly continuous derivatives of this manifold in respect to x; (3) the existence of a limited and uniformly continuous first derivative of this manifold in respect to μ."

2771:

Vasil'eva, A. B. On repeated differentiation with respect to the parameter of solutions of systems of ordinary differential equations with a small parameter in the derivative. Mat. Sb. (N.S.) **48 (90)** (1959), 311–334. (Russian)

Consider the system (1) $\mu dz/dt = F(z, x, t)$, $dx/dt = f(z, x, t)$, (2) $z(0)=z^0$, $x(0)=x^0$, where μ is a small positive parameter. Under certain assumptions A. Tihonov proved [Mat. Sb. (N.S.) **22 (64)** (1948), 193–204; MR **9**, 588] that the solutions $z(t, \mu)$ and $x(t, \mu)$ of (1) and (2) converge, as $\mu\to0+$, to the solution of the reduced system (3) $0=F(z, x, t)$, $dx/dt = f(z, x, t)$ corresponding to the stable root $z=\varphi(x, t)$ of the equation $F(z, x, t)=0$ and satisfying $x(0)=x^0$. This convergence holds in a certain interval $\langle 0, T\rangle$ with T independent of μ. Assuming that the function F has continuous derivatives of the $(n+1)$th order and f of the nth order, the author proves that the nth derivatives of the solutions $z(t, \mu)$ and $x(t, \mu)$ with respect to μ have limits, as $\mu\to0+$, in the interval $(0, T\rangle$, and these limits satisfy a system which is obtained from (1) on differentiating n times with respect to μ and putting $\mu=0$.

M. Zlámal (Brno)

2772:

Vasil'eva, A. B. Construction of uniform approximations to solutions of systems of differential equations with a small parameter in the highest derivative. Mat. Sb. (N.S.) **50** (**92**) (1960), 43–58. (Russian)

This is the continuation of earlier investigations by the author [Mat. Sb. (N.S.) **31** (**73**) (1952), 587–644; Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **9** (1954), no. 3, 29–40; no. 6, 39–46; Dokl. Akad. Nauk SSSR **119** (1958), 9–11; MR **14**, 1086; **16**, 362; **20** #4061; and #2771 above]. The topic is the study of the conditions under which the solution $z(t, \mu)$, $x(t, \mu)$ of

$$(1) \qquad \mu\dot{z} = F(z, x, t), \quad \dot{x} = f(z, x, t),$$

tends, as $\mu \to 0$, to a solution of the degenerate system

$$(2) \qquad F(\bar{z}, \bar{x}, t) = 0, \quad \dot{\bar{x}} = f(\bar{z}, \bar{x}, t).$$

It is assumed that f, F are uniformly continuous in t on a finite interval $[0, T]$. In the earlier work it was shown that the solution of (1), with initial values z^0, x^0 at $t=0$, tends to a solution of (2) with initial value x^0 for $\bar{x}$. The initial values for the nth derivatives have to be specified and described. In the present paper the author obtains asymptotic expansions relative to $\mu > 0$ and small. [Additional references: Tihonov, Mat. Sb. (N.S.) **22** (**64**) (1948), 193–204; **31** (**73**) (1952), 575–586; MR **9**, 588; **14**, 1085; Volosov, ibid. **31** (**73**) (1952), 645–674; MR **14**, 1086; Pontryagin, Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 605–626; MR **20** #1029a; Višik and Lyusternik, Uspehi Mat. Nauk (N.S.) **12** (1957), no. 5 (77), 3–122; MR **20** #2539].
S. Lefschetz (Princeton, N.J.)

2773:

Erdélyi, A. Asymptotic solutions of differential equations with transition points or singularities. J. Mathematical Phys. **1** (1960), 16–26.

The differential equations have the form

$$\frac{d^2y}{dx^2} + [\lambda^2 p(x) + r(x, \lambda)]y = 0,$$

in which λ is a large positive parameter, and x ranges over an interval (a, b) of the real axis, finite or infinite. Asymptotic forms are found for the solutions, under precise conditions, in the cases when (i) $p(x)$ is free from singularities and has just one zero in (a, b), (ii) $p(x)$ is free from zeros and has just one singularity, a simple pole, in (a, b). The function $r(x, \lambda)$ is subject to explicit conditions, and in the second case is permitted to have a double pole at the singularity of $p(x)$. The asymptotic solutions are uniformly valid with respect to x.

Applications are made to Bessel functions, Hermite polynomials and Laguerre polynomials of large orders, and the relation of the present results to those of other writers is discussed.
F. W. J. Olver (Teddington)

2774:

Halanaĭ, A. [Halanay, A.]. Periodic and almost-periodic solutions of systems of differential equations with lagging argument. Rev. Math. Pures Appl. **4** (1959), 685–691. (Russian)

This paper generalizes well-known results for differential equations to differential-difference equations. Lemma: Consider the system (1) $\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) + f(t)$, where x is an n-vector, $A(t)$, $B(t)$, $f(t)$ are bounded for $t \geq 0$, and τ is a real constant. If the solution $x=0$ of the homogeneous equation (1) is uniformly asymptotically stable, then every solution of the nonhomogeneous equation (1) is bounded in $[0, \infty)$. This lemma is proved by using Liapunov functionals. Easy consequences of this lemma are the following results: (I) if A, B, f are periodic of period ω, then (1) has an almost periodic solution of period ω; (II) if A, B, f are almost periodic, then (1) has an almost periodic solution. The results are generalized to nonlinear differential-difference equations. See S. N. Šimanov [Dokl. Akad. Nauk SSSR **125** (1959), 1203–1206; MR **21** #5051] for the case where A, B are constant.
J. K. Hale (Baltimore, Md.)

2775:

Bellman, Richard; Cooke, Kenneth L. Asymptotic behavior of solutions of differential-difference equations. Mem. Amer. Math. Soc. No. 35, 91 pp. (1959).

The title of this memoir is itself a description of its content. The authors are primarily concerned with the asymptotic behavior of solutions of the scalar equations

$$(1) \qquad u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t-\omega) = 0,$$

where $a(t) \to 0$ and $b(t) \to 0$. The paper begins with a review of already known results relative to the equation with constant coefficients,

$$(2) \qquad u'(t) + a_0 u(t) + b_0 u(t-\omega) = 0.$$

As equation (1) is a generalization of equation (2), so are the results in the paper a generalization of known results for (2).

The characteristic equation $s + a_0 + b_0 e^{-\omega s} = 0$ has roots which lie asymptotically along the curve $\operatorname{Re}(s) + \log|s| = \log|b_0|$. The authors show that to each root of this equation of multiplicity m, there are associated m solutions of known asymptotic form. There is one real root, or a conjugate pair of complex roots, having a real part which exceeds the real parts of all other roots. This root (or each root of the conjugate pair) is called the principal root. After definitions there follow eight rather lengthy lemmas. There then follows a proof of an asymptotic form for a solution corresponding to a real and simple principal root. This is done first only under the assumption that $a(t) \to 0$ and $b(t) \to 0$. There follows, however, a detailed discussion for the case that $a(t)$ and $b(t)$ have asymptotic power series expansions. Free use is made of integral equation representation and of "Liouville" transformations. There then follows a proof of the asymptotic representation of solutions corresponding to any simple root. The problem of multiple roots is treated by first discussing methods for the similar problem for ordinary differential equations, particularly a discussion of the "Liouville" transformation in general. The actual theorems proved are for double roots.

As would be expected, much detailed analysis is necessary for the proof of theorems. The theorems themselves are long in statement. The analysis however is fundamentally simple and not hard to follow if studied in detail. As the authors state, many results can be extended to vector-matrix equations.
T. Fort (Coral Gables, Fla.)

2776:

Korobeĭnik, Yu. F. Investigation of differential equations of infinite order with polynomial coefficients by means of operator equations of integral type. Mat. Sb. (N.S.) **49 (91)** (1959), 191–206. (Russian)

The equations are of the form

$$(1.1)\qquad y(x)+\sum_{k=1}^{\infty} y^{(k)}(x)\sum_{n=0}^{k-1} a_n^{(k)}x^n = g(x),$$

where $g(x)$ is entire and $F(x,t)=\sum_{s=1}^{\infty}\sum_{n=0}^{s-1} a_n^{(s)}x^n s!/t^s$ is analytic in T_{R_0}: $|x|\leq R_0$, $|t|\geq R_1=\delta R_0$. (F is then analytic in T_R: $|x|\leq R$, $|t|\geq \delta R$ for arbitrary $R\geq R_0$.) (1.1) is equivalent to $y(x)+L[y]=g(x)$ where $L[y]=(2\pi i)^{-1}\int (y(x+t)F(x,t)/t)\,dt$, taken over the contour $|t|=\delta R$ and with $|x|\leq R$, $R\geq R_0$. $y(x)$ an entire function implies $L[y]$ entire. Let $\alpha(r)=\max|F(x,t)|$ ($|x|\leq r$, $|t|=\delta r$, $r\geq R_0$), and let $f(r)$ be a non-decreasing function on $[R_0,\infty)$ satisfying (1.4) $\limsup_{r\to\infty} f(\beta r)\alpha(r)/f(r)=g_1<1$ with $\beta=1+\delta$. Let S be the family of entire functions $\{\varphi\}$ for which $\limsup_{r\to\infty} M_r[\varphi]/f(r)<\infty$ ($M_r[\varphi]=\max_{|x|\leq r}|\varphi(x)|$). Let R_2 be a sufficiently large number (with $R_2>\max(R_0,R_1)$). Then the linear set S becomes a Banach space with norm $\|\varphi\|=\sup_{r\geq R_2} M_r[\varphi]/f(r)$. (Moreover, every Cauchy sequence in S converges uniformly to its norm-limit on every bounded set.) Operator L maps S into S and is a contraction operator, so the Banach fixed point theorem applies to give the result: (1.1) has a unique solution $y(x)\in S$ for each $g(x)\in S$.

The iterative process of the fixed point theorem is sometimes not convenient for calculating the solution, so the author turns to another representation of the solution. Assume that $f(r)$ satisfies $\lim_{r\to\infty} f(r)/\log r=\infty$ in addition to (1.4), and that to $g(x)$ corresponds $\theta>1$ such that $g(\theta x)\in S$. Then (1.1) has a polynomial solution $Z_n(x)$ ($\in S$) when the right side is x^n; and when $g(x)$ is put back, the unique solution $y(x)$ (in S) of (1.1) is given by $y(x)=\sum_{k=0}^{\infty} c_k Z_k(x)$ ($c_k=g^{(k)}(0)/k!$), convergent in norm, and also uniformly convergent on every bounded set. There are further results which are too special to state in limited space.

I. M. Sheffer (University Park, Pa.)

2777:

Kisyński, J. Sur les équations différentielles dans les espaces de Banach. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 381–385. (Russian summary, unbound insert)

Let $f(t,x)$ be a continuous function from $[0,a]\times E$ into E, where $a>0$, and E is a Banach space. The author considers the differential equation (*) $x'=f(t,x)$, $x(0)=x_0$, and proves that a unique solution exists on $[0,a]$, and may be obtained by the method of successive approximations, with an arbitrary element of E as first approximation, provided the following two hypotheses hold. (1) For every $r>0$, there exists a real-valued function $w_r(t,u)$ defined on $(0,a]\times[0,2r]$, measurable in t for fixed u, continuous and non-decreasing in u for fixed t, $w_r(t,0)=0$ for almost all $t\in[0,a]$, $\int_\varepsilon^a w_r(t,2r)dt<+\infty$, such that $u(t)\equiv 0$ is the unique continuous solution of $u(t)=\int_0^t w_r(\tau,u(\tau))d\tau$, $u'(0)=0$, on $[0,\varepsilon]$ when $0<\varepsilon<a$, and such that $\|f(t,x)-f(t,y)\|\leq w_r(t,\|x-y\|)$, $0<t\leq a$, $\|x\|$, $\|y\|\leq r$. (2) If $M(t,r)=\sup\|f(t,x)\|$, $\|x\|\leq r$, and $\rho_0>0$ is arbitrary, there exists a continuous function defined on $[0,a]$ satisfying $\rho(t)=\int_0^t M(\tau,\rho(\tau))d\tau+\rho_0$.

The author mentions that (1) is needed not only to show the uniqueness, and the convergence of successive approximations, but also for the existence of the solution. There is a second theorem which states, under the same hypotheses, that the solution of (*) is a continuous function of the initial condition x_0.

A. Stokes (Baltimore, Md.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 2745, 2957, 3001, B3090, B3111.

2778:

Franklin, Philip. ★Differential equations for engineers. Dover Publications, Inc., New York, 1960. vii+299 pp. Paperbound: \$1.65.

This edition is an unabridged and unaltered republication of the first edition of the work first published in 1933 under the title *Differential equations for electrical engineers* [Wiley, New York].

The chapter headings are as follows: (1) Complex numbers; (2) Average values and Fourier series; (3) Linear differential equations with constant coefficients; (4) Partial derivatives and partial differential equations; (5) The physical meaning of certain partial differential equations; (6) Solutions of partial differential equations satisfying given boundary values; (7) Analytic functions; (8) Convergence of Fourier series.

2779:

Weinstein, Alexander. On a singular differential operator. Ann. Mat. Pura Appl. (4) **49** (1960), 359–365.

For operators of the form

$$D_k(u)=\frac{d^2u}{dt^2}+\frac{k}{t}\frac{du}{dt}$$

certain recursion formulas had previously been established by the author [Proc. Symposia Appl. Math., Vol. V, pp. 137–147, McGraw-Hill, New York, 1954; Ann. Mat. Pura Appl. (4) **43** (1957), 325–340; MR **16**, 137; **19**, 656]. Let $X(u)$ be a linear operator and consider solutions of $D_k(u)=X(u)$. The recursion formulas are extended to cover more general operators and, further, an application is given which generalizes a classical formula for confluent hypergeometric functions. Finally the author establishes the principle of the Kelvin transformation for equations of the form

$$\frac{\partial^2 v}{\partial r^2}+\frac{k}{r}\frac{\partial v}{\partial r}=\frac{1}{r^2}\Phi(v),$$

where Φ is linear, independent of r and vanishes for $v\equiv 0$. This contains the result of Huber [Proc. Conference on Differential Equations, pp. 147–155, Univ. of Maryland Bookstore, College Park, Md., 1956; MR **18**, 650] as well as the standard theorem for harmonic functions.

M. H. Protter (Berkeley, Calif.)

2780:

Salahitdinov, M. On the method of characteristics for simultaneous first order partial differential equations with non-integrable relations. Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. **1958**, no. 3, 51–65. (Russian. Uzbek summary)

The method of I. S. Aržanyh [Uspehi Mat. Nauk **9** (1954), no. 3 (61), 119–125; MR **16**, 479] is extended to systems of the form

$$F_\kappa(x_1, \cdots, x_n, z, p_1, \cdots, p_n, \lambda_1, \cdots, \lambda_m) = 0 \qquad (\kappa = 1, \cdots, r),$$

$$f_i(x_1, \cdots, x_n, z, p_1, \cdots, p_n, \lambda_1, \cdots, \lambda_m) = 0 \qquad (i = 1, \cdots, m),$$

$$p_\nu = \partial z/\partial x_\nu \qquad (\nu = 1, \cdots, n),$$

where the λ's are parameters depending on $x_1, \cdots, x_n, z, p_1, \cdots, p_n$, and the f's are non-integrable relations. Under the assumption that the F's satisfy certain compatibility conditions, a Cauchy problem may be posed leading to a theory of characteristics for the system. The procedure employed by Aržanyh consists in writing the equations of the characteristics in parametric form: $dx_\nu = X_\nu dt$, $dz = Zdt$, $dp_\nu = P_\nu dt$, $d\lambda_j = \Lambda_j dt$ and then computing X_ν, Z, P_ν, Λ_j. The equations thus obtained for the differentials are shown to be integrable. *R. N. Goss* (San Diego, Calif.)

2781:

Protter, M. H. Lower bounds for the first eigenvalue of elliptic equations. Ann. of Math. (2) **71** (1960), 423–444.

The author considers the eigenvalue problem whose first eigenvalue is characterized by the minimum principle

$$\lambda_1 = \min_\varphi \iint_{\mathscr{D}} \sum_{i,j=1}^{n} a^{ij} \frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j} dV \Big/ \iint_{\mathscr{D}} \varphi^2 dV$$

defined on a simply connected n-dimensional region $\mathscr{D}$ with boundary Γ. The minimum is taken among all piecewise smooth functions φ which vanish on Γ. It is assumed that

$$C_0 \sum_{i=1}^{n} \xi_i^2 \leqq \sum_{i,j=1}^{n} a^{ij}\xi_i\xi_j \leqq C_1 \sum_{i=1}^{n} \xi_i^2$$

for all real quantities ξ_i, where C_0 and C_1 are positive constants.

It follows from the addition of a divergence term of the type

$$\iint_{\mathscr{D}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(p^i\varphi^2) dV$$

(which by Green's theorem is zero) to the numerator of the Rayleigh quotient and regrouping, that

$$\lambda_1 \geqq \min[p_{,i}{}^i - a_{jl}p^jp^l].$$

Here a_{jl} is the inverse of a^{jl}, and p^j is an arbitrary vector.

In the particular case of the ordinary membrane equation the author develops an algorithm which yields an increasing sequence of lower bounds for λ_1. He also compares his method with that of symmetrization, and concludes by extending the method to treat eigenvalue problems for elliptic equations of arbitrary order. *L. E. Payne* (Newcastle-upon-Tyne)

2782:

Mangeron, D. I. Problème des spectres pour les systèmes différentiels réductibles. Bul. Inst. Politech. Iaşi **4** (1949), 441–445.

The problems considered are the n-dimensional generalizations of problems relative to partial differential equations of the type

$$\frac{\partial^{m+n}u}{\partial x^m \partial y^n} - \lambda A(x, y)u = 0,$$

with various boundary conditions on the surface of a rectangle R, with sides parallel to the coordinate axes. The terminology "problèmes reductibles" was introduced by M. Picone [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1** (1940), 642–655; MR **3**, 44] to designate boundary value problems of this type for which the translation into integral equations depends only upon the construction of Green's functions for similar problems in a smaller number of independent variables [see also the author, C. R. Acad. Sci. Paris **204** (1937), 94–96, 544–547, 1022–1024; Mathematica. Cluj **14** (1938), 31–35]. *J. B. Diaz* (College Park, Md.)

2783:

Wolf, František. On singular partial differential boundary problems. Ann. Mat. Pura Appl. (4) **49** (1960), 167–179.

The author studies the essential spectrum produced by a second order elliptic partial differential operator when the ellipticity breaks down on part of the boundary. The operator is

$$l \equiv -\sum_{i,k}^{1,\cdots,n} \frac{\partial}{\partial s_i} a_{ik} \frac{\partial}{\partial s_k} + \sum_i b_i \frac{\partial}{\partial s_i} + c,$$

where all (complex) coefficients are bounded in a domain G and the a_{ik} are once boundedly differentiable. It is assumed that $\left|\sum (a_{ik}+\bar{a}_{ki})\bar{\xi}_i\xi_k\right| \geqq \sigma(s)\sum|\xi_i|^2$, $\sigma(s)$ being positive inside G but being allowed to vanish at some parts of the boundary. The domain of l is the set of infinitely differentiable functions x which vanish near one part T_0 of the boundary ∂G and satisfy $F(x, y) = (lx, y)$ for all $y \in C^2$, where

$$F(x, y) \equiv \left(\sum a_{ik}\frac{\partial x}{\partial s_i}, \frac{\partial y}{\partial s_i}\right) + \left(\sum b_i \frac{\partial x}{\partial s_i}, y\right) + (cx, y) + \int_{\partial G} A\bar{x}\cdot\bar{\bar{y}}d\bar{s},$$

(x, y) is the $L^2(G)$ inner product, and A is a bounded linear operator in $L^2(\partial G)$. It is assumed that $2\operatorname{Re} F(x, x) + \lambda(x, x) \geqq [x, x]$ for all $x \in C^2$ and λ sufficiently large, where

$$[x, y] = \tfrac{1}{2}\int \sum (a_{ik}+\bar{a}_{ki})\frac{\partial x}{\partial s_k}\frac{\partial \bar{y}}{\partial s_i} ds + \tfrac{1}{2}\int (\operatorname{Re} c + |\operatorname{Re} c|)x\bar{y}dx + (x, y)$$

(sufficient conditions for this are obtained). The Lax-Milgram lemma [P. D. Lax and A. N. Milgram, *Contributions to the theory of partial differential equations*, pp. 167–190, Princeton Univ. Press, Princeton, N.J., 1954; MR **16**, 709] is then employed to define an extension $L+\lambda$ of $l+\lambda$ in the following way: If $\mathscr{M}$ is the closure of C^2 with respect to the norm $[x, x]$, $L+\lambda$ is the operator C of $\mathscr{M} \to L^2(G)$ defined by $(x, y) = F(C^{-1}x, y) + \lambda(C^{-1}x, y)$.

Next a surface Σ is introduced in a regular part of G. Another extension M of l is defined in a similar way, with the exception that Σ is now considered part of ∂G and functions in the domain of M may have jump discontinuities across Σ, but must satisfy regular boundary

conditions on both sides. The main theorem states that $(L+\lambda I)^{-1}-(M+\lambda I)^{-1}$ is a compact operator. Several applications and generalizations are given.
M. Schechter (New York)

2784:

Bochenek, K. Analytic solutions of the eikonal equation. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 657–660. (Russian summary, unbound insert)

The author studies the partial differential equation $(\nabla L)^2=1$ in two dimensions, with a view to applying the results in asymptotic expansions of the solutions of the Helmholtz equation. Let $v(\alpha)$ be the unit vector making angle α with a fixed direction, and let $b(\alpha)$ be a real scalar function of period 2π, analytic on the real axis. Then $v(\alpha)\cdot x+b(\alpha)$ is a one-parameter family of solutions (x is the position vector, and $\cdot$ indicates a scalar product). The author forms solutions of $(\nabla L)^2=1$ by forming envelopes of such one-parameter families, and discusses these solutions. *A. Erdélyi* (Pasadena, Calif.)

2785:

Henrici, Peter. Complete systems of solutions for a class of singular elliptic partial differential equations. Boundary problems in differential equations, pp. 19–34. Univ. of Wisconsin Press, Madison, 1960.

The author solves the Cauchy problem: $L_\nu u=u_{xx}+u_{yy}+2\nu y^{-1}u_y+au_x+cu=0$; $u(x,0)=f(x)$, $u_y(x,0)=0$ for $u=u(x,y)$, assuming that a and c are analytic functions of x alone. Complex values of x and y are admitted, and f is also assumed to be analytic. The solution, u_0, for $\nu=0$ is represented in terms of the Riemann function of the equation $L_0u=0$, and the solution for $\nu>0$ is obtained by means of the formula

$$u(x,y)=\int_0^\pi u_0(x,y\cos\theta)(\sin\theta)^{2\nu-1}d\theta\Big/\int_0^\pi(\sin\theta)^{2\nu-1}d\theta$$

and then expressed in terms of the Riemann function of $L_0u=0$. The class of solutions so obtained is investigated and is shown to consist, roughly speaking, of all solutions analytic at some point of the x axis (the singular line of the differential equation). It is also shown that the sequence of solutions corresponding to $f(x)=x^m$ ($m=0,1,2,\cdots$) forms a complete system, in the sense that every solution for a region R can be approximated arbitrarily closely, and uniformly over compact subdomains, by a linear combination of solutions of the sequence; and a criterion for complete systems is obtained.

As an example for the general formula, the case $a=0$, c a positive constant, for which the Riemann function is explicitly known, is considered, and a complete system is obtained also for the differential equation $u_{xx}+u_{yy}+2\mu u_x+2\nu u_y+k^2u=0$. *A. Erdélyi* (Pasadena, Calif.)

2786:

Hirasawa, Yoshikazu. Principally linear partial differential equations of elliptic type. Funkcial Ekvac. **2** (1959), 33–94. (Esperanto summary)

The author studies equations of the form $Lu=f(x,u,\partial_xu)$ where L is a linear elliptic operator of the second order. The operator L is first replaced by a generalized operator $\mathscr{L}$ in the sense of Gevrey [Ann. Sci. École Norm. Sup. (3) **52** (1935), 39–108] and an elementary solution is obtained for $\mathscr{L}$. Comparison and uniqueness theorems are given which extend the results of Tokui Satō [Compositio Math. **12** (1954), 157–177; MR **17**, 474] obtained for the case of two variables and L replaced by the Laplacian.

Existence theorems, solution of the first boundary problem, estimates for the derivatives and a suitable generalization of Harnack theorems are obtained. The methods employ Schauder estimates and extend the work of Nagumo [Osaka Math. J. **6** (1954), 207–229; MR **16**, 1116] from strict to generalized operators.
M. H. Protter (Berkeley, Calif.)

2787:

Schechter, Martin. A free boundary problem for pseudo-analytic functions. Proc. Amer. Math. Soc. **10** (1959), 881–887.

Let Γ be a simple closed curve in the plane and suppose that (1) $\varphi(x,y)$ satisfies $\Delta\varphi=g(x,y,\varphi_x,\varphi_y)$ in a domain D except at the origin P which is an interior point, (2) φ has continuous first derivatives in $\bar{D}-P$ and $\varphi_x^2+\varphi_y^2=1$ on Γ, (3) the normal derivative of φ vanishes on Γ and (4) $(\varphi_x^2+\varphi_y^2)(x^2+y^2)\to\lambda^2\neq0$ as $(x,y)\to P$. Under suitable assumptions on g it is shown that there exist Γ, φ, λ_0 satisfying (1)–(4) for $0<\lambda<\lambda_0$. The procedure consists of translating the problem into a corresponding (more general) one in terms of pseudo-analytic functions and employing the methods of this theory, as originated by Bers [*Theory of pseudo-analytic functions*, New York University, New York, 1953; MR **15**, 211], to solve the appropriate integral equation.
M. H. Protter (Berkeley, Calif.)

2788:

Il'in, A. M. Degenerate elliptic and parabolic equations. Mat. Sb. (N.S.) **50 (92)** (1960), 443–498. (Russian)

The article deals with the first boundary problem for degenerate elliptic and parabolic second-order equations. For the statement of the principal results, let us say that the function ϑ belongs to class $C^{(\kappa,\alpha)}$ if ϑ has derivatives of order κ which satisfy a Hölder condition with exponent α. Consider, in the $(n+1)$-dimensional region D with boundary S, the equation

$$(1)\quad L(u)\equiv f\frac{\partial^2u}{\partial x_0^2}+\sum_{i,j=1}^n a_{ij}\frac{\partial^2u}{\partial x_i\partial x_j}+\sum_{i=0}^n b_i\frac{\partial u}{\partial x_i}+cu=F.$$

Assume that $c\leq0$, $\sum_{i,j=1}^n a_{ij}(x)\alpha_i\alpha_j\geq\mu\sum_{i=1}^n\alpha_i^2$, $\mu>0$, $f(x)\geq0$ in D and $f=0$ on the arbitrary set $D_0\subset D$; the equation's coefficients and F belong to $C^{(\kappa,\alpha)}$, $\kappa\geq2$, while S belongs to class $C^{(\kappa+2,\alpha)}$. It is proved that if D_0 does not intersect S, and if at the points of D_0 the condition

$$(2)\qquad \tfrac{1}{2}\kappa(\kappa+1)\frac{\partial^2f}{\partial x_0^2}+(\kappa+1)\frac{\partial b_0}{\partial x_0}+c<0$$

is fulfilled, then there exists in D a unique solution of (1) equal to zero on S and belonging to $C^{(\kappa,1)}$. An example is adduced showing that condition (2) is essential. Analogous theorems are established also in the case that D_0 intersects S. Conditions on the coefficients and S are given under which points of S belonging to D_0 are excepted from the

boundary condition. The solution of the boundary value problem for equation (1) is obtained as the limit for $\varepsilon \to 0$ of solutions $u_\varepsilon(x)$ of the Dirichlet problem for the elliptic equation

$$(f+\varepsilon)\frac{\partial^2 u}{\partial x_0^2}+\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{=0}^{n} b_i \frac{\partial u}{\partial x_i}+cu=F, \quad \varepsilon>0,$$

with the condition $u_\varepsilon|_S=0$, by use of a priori estimates of the derivatives of u_ε, uniform with respect to ε.

There is also investigated the first boundary problem, in the cylinder $Q=D\times[0, T]$, for an equation of the form $\partial u/\partial t=Lu+F$ with coefficients depending on t and x. Here in the setting of the problem the set Q_0 where $f=0$ may be an arbitrary set interior to Q, no condition such as (2) being imposed. *O. A. Oleĭnik* (Moscow)

2789:

Fichera, Gaetano. On a unified theory of boundary value problems for elliptic-parabolic equations of second order. Boundary problems in differential equations, pp. 97–120. Univ. of Wisconsin Press, Madison, 1960.

In this paper the author describes a unified theory of boundary value problems for second order linear equations of elliptic-parabolic type, including the classical cases of totally elliptic equations, the heat equation, and first order equations. The methods used are elegant, the results are interesting if not yet best possible, and the author's claim that the theory "opens up many attractive fields for further research" seems justified. The results for a first order equation are overlapped by those of K. O. Friedrichs [Comm. Pure Appl. Math. **11** (1958), 333–418; MR **20** #7147], although the latter is principally interested in systems rather than single equations.

Let A be a domain in X^r, r-dimensional real Cartesian space, whose boundary Σ is composed of regular $r-1$-cells. Let $L(u)\equiv a^{ij}u_{x_ix_j}+b^iu_{x_i}+cu$, $a^{ij}(x)=a^{ji}(x)$, where a^{ij}, b^i, c belong to $C^2(B)$, $C^1(B)$, and $C^0(B)$ respectively, and $B\supset\bar{A}$. Assuming $a^{ij}(x)\lambda_i\lambda_j\geqq 0$ for all real $\lambda=(\lambda_1,\cdots,\lambda_n)\neq 0$ and $x\in B$, L is said to be a positive elliptic-parabolic operator in B. Let Σ' denote the points where the boundary is characteristic, and $\Sigma^{(1)}$ the points of Σ' where $b(x)=(b^i-a_{x_j}{}^{ij})n_i$ is non-negative, $\Sigma^{(2)}=\Sigma'-\Sigma^{(1)}$ and $\Sigma^{(3)}=\Sigma-\Sigma'$. Finally let $\Sigma_D{}^{(3)}$ and $\Sigma_N{}^{(3)}$ be disjoint sets such that $\Sigma^{(3)}=\Sigma_D{}^{(3)}\cup\Sigma_N{}^{(3)}$. For f, g, h given functions defined respectively on A, $\Sigma^{(2)}\cup\Sigma_D{}^{(3)}$ and $\Sigma_N{}^{(3)}$, the author's principal interest is the boundary problem: $Lu=f$ in A; $u=g$ on $\Sigma^{(2)}\cup\Sigma_D{}^{(3)}$, $a^{ij}(x)u_{x_i}n_j=h$ on $\Sigma_N{}^{(3)}$. L_p a priori estimates are established, and also certain maximum principles. These yield uniqueness theorems in the usual way. In a later section a criterion is formulated for existence of solutions to certain functional equations in Banach spaces and applied to prove existence of weak solutions of elliptic-parabolic equations. *A. N. Milgram* (Minneapolis, Minn.)

2790:

Gehring, F. W. The boundary behavior and uniqueness of solutions of the heat equation. Trans. Amer. Math. Soc. **94** (1960), 337–364.

Let x, t be real variables, let H be the class of solutions of $u_{xx}=u_t$ in $0<t<c$, $c>0$; let H^+ be the subset of non-negative solutions, and let H^Δ be the set of $u=u_1-u_2$ where u_1, u_2 belong to H^+. Widder proved that $u\in H^\Delta$ if and only if it can be represented as

$$u(x,t)=\int_{-\infty}^{\infty} k(x-y,t)d\alpha(y)$$

in $0<t<c$, where $k(x,t)=(4\pi t)^{-1/2}\exp(-x^2/4t)$ and $\alpha(x)$ is of bounded variation in bounded intervals. α may be normalized by $2\alpha(x)=\alpha(x+0)+\alpha(x-0)$. Set $S(a,x_0)=\{(x,t)\colon |x-x_0|<at,\ t>0\}$, $D\alpha(x_0)=\lim_{h\to 0}[\alpha(x_0+h)-\alpha(x_0-h)]/2h$. The author proves Fatou type theorems, some converses, and also some uniqueness theorems for functions of the classes defined above. A few typical results will be quoted here. (1) If $u\in H^\Delta$ and $D\alpha(x_0)=A$, then for any $a>0$, $u(x,t)\to A$ as $(x,t)\to(x_0,0)$ in $S(a,x_0)$. (2) Suppose $u\in H^\Delta$, $c=\infty$, and assume that $[\alpha(x_0+x)-\alpha(x_0-x)]/2x\to A$ as $x\to\infty$. Then $u(x_0,t)\to A$ as $t\to\infty$. (3) If $u\in H^+$ and if, for some a, $u(x_0+at)\to A$ as $t\to 0^+$, then $D\alpha(x_0)=A$. (4) Suppose $u\in H^+$, $c=\infty$ and assume that $u(x_0,t)\to A$ as $t\to\infty$. Then $[\alpha(x_0+x)-\alpha(x_0-x)]/2x\to A$ as $x\to\infty$. (5) Suppose $u\in H^+$ and $c=\infty$. If for some x_0, $u(x_0,t)\to A$ as $t\to\infty$, then for each a, $u(x,t)\to A$ as $t\to\infty$, uniformly in x, $|x-x_0|<a$. (6) If $u\in H^\Delta$ and $\lim_{t\to 0+}u(x,t)<\infty$ for all x for which this limit exists, and if $\lim_{t\to 0+}u(x,t)\leqq A$ for almost all x, then $u(x,t)\leqq A$ in $0<t<c$. *A. Friedman* (Minneapolis, Minn.)

2791:

Pogożel'skiĭ, V. [Pogorzelski, W.]. Investigation of integrals of a parabolic equation and of boundary problems in an unbounded domain. Mat. Sb. (N.S.) **47** (**89**) (1959), 397–430. (Russian)

Consider the equation

$$(1)\quad \Psi u\equiv\sum_{i,j=1}^{n} a_{ij}(x,t)u_{,ij}+\sum_{i=1}^{n} b_i(x,t)u_{,i}+c(x,t)u-u_{,t}=0$$

for $(x,t)\in\Omega\times[0,T]=R$, where $\Omega\subseteq E^n$. Assume that Ψ is uniformly parabolic, the a_{ij} are uniformly Hölder continuous with respect to x and t, and the b_i and c are bounded continuous functions of (x,t) which are Hölder continuous with respect to x in R. For $\Omega=E^n$ the author proves the existence of a fundamental solution of (1). This extends his earlier results for Ω bounded [Ricerche Mat. **5** (1956), 25–57; MR **18**, 47]. {A slightly more general result has been obtained by the reviewer [Illinois J. Math. **3** (1959), 580–619; MR **21** #6480].} Using the fundamental solution the author solves linear and nonlinear third boundary problems for (1) in case Ω is a region whose boundary consists of a finite number of disjoint $(n-1)$-dimensional surfaces which are either closed or unbounded and satisfy a Liapunov condition. In particular, he extends his earlier results for bounded Ω [Ann. Polon. Math. **4** (1957), 61–92, 110–126; **21** #3656a, b]. *D. G. Aronson* (Minneapolis, Minn.)

2792:

Vol'pert, A. I. Calculation of the index of Dirichlet's problem. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 1042–1044. (Ukrainian. Russian and English summaries)

In this paper the author considers a homogeneous Dirichlet problem for the elliptic system

$$\sum A_{kl}(z)\frac{\partial^{k+l}u}{\partial x^k\partial y^l}=0,$$

where $k+l\leq 2$, A_{kl} are square matrices and the unknown u is a vector of functions. Set

$$R_n(z) = (2\pi i)^{-1}\int_\gamma (\sum A_{kl}(z)\alpha_l)^{-1}\alpha^n d\alpha \quad (n = 0, \pm 1),$$

where γ is a suitably chosen contour in the plane $\operatorname{Im}\alpha>0$. Let k and k^* denote the number of linearly independent solutions of the problem and its adjoint, respectively. The author calculates the index $k-k^*$ by means of the minors of the matrix $R(z)=(R_{3-i-j}(z))$ $(i, j=1, 2)$. [See Ya. B. Lopatinskiĭ, Ukrain. Mat. Ž. **5** (1953), 123–151; MR **17**, 494; and A. I. Volpert, Dokl. Akad. Nauk SSSR **114** (1957), 462–464; MR **20** #5960.]

A. N. Milgram (Minneapolis, Minn.)

2793:

Schechter, Martin. Elliptic problems in which the boundary conditions do not form a normal set. Bull. Amer. Math. Soc. **66** (1960), 84–85.

This note announces an extension of some known results on elliptic partial differential equations to the case in which the boundary conditions do not form a normal set.

K. T. Smith (Madison, Wis.)

2794:

Slobodeckiĭ, L. N. Estimates in L_2 for solutions of linear elliptic and parabolic systems. I. Estimates of solutions of an elliptic system. Vestnik Leningrad. Univ. **15** (1960), no. 7, 28–47. (Russian. English summary)

Actually this paper is concerned only with elliptic equations, and supplies the proof of results stated previously by the author [Dokl. Akad. Nauk SSSR **123** (1958), 616–619; MR **21** #5061]. A later paper is promised for establishing analogous results for parabolic equations.

A. S. Householder (Oak Ridge, Tenn.)

2795:

Lopatins'kiĭ, Ya. B. The behavior in infinity of solutions of a system of differential equations of the elliptic type. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 931–935. (Ukrainian. Russian and English summaries)

Author's summary: "Let

$$A\left(\frac{\partial}{\partial x}\right) = \sum_{k_1+\cdots+k_n=s} A_{k_1\cdots k_n}\frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}},$$

$$B\left(x, \frac{\partial}{\partial x}\right) = \sum_{k_1+\cdots+k_n\leq s} B_{k_1\cdots k_n}(x)\frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}$$

$$(x = (x_1, \cdots, x_n)),$$

where $A_{k_1\ldots k_n}$ are $p\times p$ constant matrices, $B_{k_1\ldots k_n}$ are $p\times p$ functional matrices. It is supposed that $\det A(\alpha)\neq 0$ for every non-zero real vector $\alpha=(\alpha_1, \cdots, \alpha_n)$, that $B_{k_1\ldots k_n}(x)$ are in the real domain $x_1^2+\cdots+x_n^2>R^2$, are sufficiently smooth and satisfy the conditions:

$$\frac{\partial^{l_1+\cdots+l_n}}{\partial x_1^{l_1}\cdots\partial x_n^{l_n}} B_{k_1\ldots k_n}(x) = O(|x|^{-\kappa-s+k_1+\cdots+k_n-l_1-\cdots-l_n})$$

$$(\kappa \geq 1, \kappa > s-2, l_1+\cdots+l_n \leq s+k_1+\cdots+k_n).$$

Let V_λ denote the additive group of solutions of equation (1) defined in the neighbourhood of infinity and satisfying the condition $u(x)=O(|x|^\lambda)$. Then, for any real λ, $\lambda'<\lambda$, the factor-group $V_\lambda/V_{\lambda'}$ is finite-dimensional."

2796:

Boyarskiĭ, B. On the first boundary value problem for elliptic systems of second order in the plane. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 565–570. (Russian. English summary)

The system of equations

$$Au_{xx}+2Bu_{xy}+Cu_{yy}+Eu_x+Fu_y+Du = g,$$

where A, B, C, E, F, D are given $n\times n$ matrices of functions and both g and the unknown u are vector functions with n components, is called elliptic if $\det|A\lambda^2+2B\lambda+C|\neq 0$, $\det A\neq 0$ for all real λ. Supposing the system given in a domain D with smooth boundary Γ in the x, y plane, the first boundary problem consists in seeking a solution with $u|_\Gamma=0$. If the system is elliptic but not strongly elliptic, then even for $n=2$ there may exist infinitely many linearly independent solutions [A. V. Bicadze, Uspehi Mat. Nauk **3** (1948), no. 6 (28), 211–212; MR **10**, 300]. For strongly elliptic systems, of course, the solution is unique. The present note is devoted to delineating a broader class for which the problem is correctly set. A homotopic classification of systems of two equations of the above type is given. Under certain conditions the Dirichlet problem is shown to be equivalent to a system of Fredholm equations. Some generalizations to systems of order $n>2$ are also given.

A. N. Milgram (Minneapolis, Minn.)

2797:

Abolinya, V. È.; Myškis, A. D. Mixed problems for quasi-linear hyperbolic systems in the plane. Mat. Sb. (N.S.) **50 (92)** (1960), 423–442. (Russian)

The present paper directly generalizes an earlier one by the same authors [Latvijas Valsts Univ. Zinātn. Raksti **20** (1958), no. 3, 87–104; MR **21** #212]. In abbreviated statement, one is to solve the system

$$(*) \qquad \partial u_i/\partial t+\lambda_i(x, t)(\partial u_i/\partial x) = F_i[u]+f_i(x, t) \qquad (i = 1, \cdots, m)$$

in $0\leq x\leq l$, $0\leq t\leq T$, with initial conditions $u_i(x, 0)=g_i(x)$ and boundary conditions

$$u_i(0, t) = \Phi_{0i}[u]+h_{0i}(t), \quad u_i(l, t) = \Phi_{li}[u]+h_{li}(t).$$

The expressions $F_i[u]$, $\Phi_{0i}[u]$, $\Phi_{li}[u]$, which represent (non-linear) operators converting vectors u in the lT-rectangle into scalar functions, contain the generalization of the previously studied system. These terms are of "Volterra type," in that the dependence on u is only for $t\leq t_0$. The method of solution, as before, is to pass to a system of integro-functional equations, with integration along the characteristics of (*), which system in turn is solved by successive approximations. Conditions are established (i) for existence and uniqueness of a continuous solution of the integro-functional system, which is called a "continuous generalized solution" of the original system; (ii) for smoothness of the continuous generalized solution, yielding a classical solution; and (iii) for existence and uniqueness of a solution of the integro-functional system in the Banach space of m-dimensional vector functions $u(x, t)$ with norm

$$\|u\| = \max_i\{\max_{\in L_i}\int_0^T|u_i(x, t)|dt+\max_t\int_0^l|u_i(x, t)|dx\},$$

$L_1, \cdots, L_m$ being given closed, possibly empty or coincident, sets on $0 \leq x \leq l$; which solution is called a "summable generalized solution" of the original system. Estimates are given for each of the types of solution, possibilities of weakening the various hypotheses are indicated, and properties of the discontinuities of the summable generalized solution are discussed.

R. N. Goss (San Diego, Calif.)

2798:

Kellogg, R. Bruce. Hyperbolic equations with multiple characteristics. Trans. Amer. Math. Soc. **93** (1959), 277–291.

L'auteur étudie le problème de Cauchy pour certaines équations linéaires hyperboliques à coefficients variables à caractéristiques multiples. Il se place dans le cas de deux variables x_1 et x_2 et considère les opérateurs de la forme

$$a+M = [D_{x_1}+\lambda_0(x)D_{x_2}]\cdots[D_{x_1}+\lambda_m(x)D_{x_2}]+M,$$

les fonctions $\lambda_0(x), \cdots, \lambda_m(x)$, $x \equiv (x_1, x_2)$, n'étant pas nécessairement distinctes et M désignant une combinaison linéaire des opérateurs déduits de a en omettant un ou plusieurs facteurs, les autres étant laissés dans le même ordre.

Le problème de Cauchy pour ces opérateurs a été étudié par A. Lax [Comm. Pure Appl. Math. **9** (1956), 135–169; MR **18**, 397] par un procédé d'itération. L'auteur applique ici les méthodes de J. Leray et L. Gårding [voyez L. Gårding, *Cauchy's problems for hyperbolic equations*, Univ. of Chicago Notes, 1957] pour résoudre le problème de Cauchy pour un grand nombre de données de Cauchy incluant des distributions, et ce, avec un minimum de conditions de dérivabilité sur les données. La majeure partie du travail est consacrée à l'étude des espaces fonctionnels et des inégalités a priori nécessaire à la mise en œuvre de la méthode précitée.

H. G. Garnir (Liège)

2799:

Pini, Bruno. Su un problema di tipo nuovo relativo alle equazioni paraboliche d'ordine superiore al secondo. Ann. Mat. Pura Appl. (4) **48** (1959), 305–332.

The equations studied are

$$(1)\qquad \frac{\partial^4 u}{\partial x^4}+\frac{\partial u}{\partial y}+\sum_{k=0}^{2} a_k(x)\frac{\partial^k u}{\partial x^k} = f(x,y)$$

inside the half-strip $a \leq x \leq b$, $y \geq 0$, the solution u of which takes given values on $y=0$ and $x=a$, x_1, x_2, b $(a<x_1<x_2<b)$, and

$$(2)\qquad \frac{\partial^4 u}{\partial x^4}-2a\frac{\partial^3 u}{\partial x^2\partial y}+\frac{\partial^2 u}{\partial y^2}+\sum_{k=0}^{2} a_k(x)\frac{\partial^k u}{\partial x^k}+b(x)\frac{\partial u}{\partial y} = f(x,y)$$

in the same half-strip, the solution of which takes on the same given values while its derivative $\partial u/\partial y$ assumes a given value on $y=0$. A uniqueness theorem is proved with the use of elementary inequalities for a very general boundary-value problem, including the above as special cases. The problem for (1) is treated as in the author's earlier work [same Ann. (4) **43** (1957), 261–297; MR **19**, 965] by first constructing the solution of $\mathscr{L}_0[u] \equiv \partial^4 u/\partial x^4 + \partial u/\partial y = f(x,y)$ from those of $\mathscr{L}_0[u]=0$. The same procedure is then used to obtain a solution for (1) from solutions of $\mathscr{L}_0[u]+\mathscr{L}_1[u]=0$, $\mathscr{L}_1[u]=\sum_{k=0}^{2} a_k(x)\,(\partial^k u/\partial x^k)$. At each stage estimates for the magnitudes of the derivatives are obtained. The treatment for (2) is carried out similarly by using initially the operator consisting of the three leading terms on the left of (2). The fundamental solutions employed in the case of (2) are sums of integrals of convolution type, and the problem is solved by Laplace transforms.

R. N. Goss (San Diego, Calif.)

2800:

Mlak, W. On a linear differential inequality of parabolic type. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 653–656. (Russian summary, unbound insert)

Let $u(x,t)$ be a continuous function in a rectangle R $(0 \leq x \leq 1,\ 0 \leq t \leq T)$, (1) $u(x,0) \leq 0$ for $0 \leq x \leq 1$, (2) $u(0,t)=u(1,t)=0$ for $0 \leq t \leq T$. It is a well-known result, that if $u(x,t)$ satisfies the inequality (3) $\partial u/\partial t \leq \partial^2 u/\partial x^2$ in R, then $u(x,t) \leq 0$ in R. The author proves that this result holds also in the case that (3) is satisfied only almost everywhere (in the Lebesgue sense) in R and that the function $u(x,t)$ fulfills the following conditions: (1) For almost all $x \in \langle 0, 1\rangle$ the function $u(x,t)$ is absolutely continuous with respect to t $(0 \leq t \leq T)$. (2) For almost all $t \in \langle 0, T\rangle$ the derivative $\partial u(x,t)/\partial x$ exists for all $x \in \langle 0, 1\rangle$ and is absolutely continuous with respect to x $(0 \leq x \leq 1)$. (3) The derivative $\partial^2 u/\partial x^2$ is summable in R, i.e., $\iint_R |\partial^2 u/\partial x^2|\,dx\,dt < +\infty$.

K. Rektorys (Prague)

2801:

Heinz, Erhard. On elliptic Monge-Ampère equations and Weyl's embedding problem. J. Analyse Math. **7** (1959), 1–52.

This article considers the question of setting up a priori estimates for the solution of the Monge-Ampère equation of elliptic type:

$$\begin{gathered} F \equiv A(x,y,z,p,q)r+2B(x,y,z,p,q)s+C(x,y,z,p,q)t \\ +(rt-s^2)-E(x,y,z,p,q) = 0 \\ (p=z_x,\ q=z_y,\ r=z_{xx},\ s=z_{xy},\ t=z_{yy}), \\ \Delta = AC-B^2+E > 0. \end{gathered}$$

The basic results are as follows. (1) If $z(x,y)$ is a solution of class $C^{3+\nu}$ $(0<\nu<1)$ of the equation $F=0$ in the region Ω and if $\bar{\Omega}$ is a closed subset of Ω, then in $\bar{\Omega}$ the second and third derivatives of the solution satisfy a Hölder condition with index ν and constant C depending only on the maximum modulus of the solution $z(x,y)$ and its derivatives up to second order, on the maximum modulus of the coefficients of the equation and of their derivatives up to second order, on the lower bound of the discriminant Δ, and on the distance of $\bar{\Omega}$ from the boundary of Ω. If the coefficients A, B, C of the equation do not depend on p and q, then a priori estimates are obtained also for the second derivatives of the solution, depending on the above-mentioned quantities with the exception, of course, of the second derivatives themselves. (2) For the second derivatives of the solution $z(u,v)$ of the Darboux equation corresponding to the line element $ds^2=Edu^2+2Fdudv+Gdv^2$ in $\bar{\Omega}$, the author obtains estimates depending on the maximum modulus of the coefficients of the line element and their derivatives up to the fourth order, on the lower bound δ_0 of the discriminant of the form ds^2, on the lower bound k_0 of the Gaussian curvature of the surface Φ with

line element ds^2 determined by the solution $z(u, v)$, and on the upper bound α_0 of the angles formed by the tangent planes of Φ with the xy plane, under the assumption that δ_0, k_0, and $\alpha_0 > 0$. This result is used to solve the problem of constructing a closed convex surface which realizes a given metric on the sphere by a line element ds^2, according to the method of H. Weyl.

The basic tool employed to obtain the a priori estimates consists of the transformation to a special isothermal parametrization u, v determined by the condition $F_t dx^2 - F_s dx dy + F_r dy^2 = \lambda(du^2 + dv^2)$. This use of isothermal coordinates makes the author's method quite similar to that of H. Lewy, who made an analogous transformation to asymptotic parameters. Isothermal coordinates have the advantage over asymptotic, that the method can also be employed for equations with non-analytic coefficients.

A. V. Pogorelov (Kharkov)

2802:

Gol'denveĭzer, A. L. Asymptotic integration of partial differential equations with boundary conditions depending on one parameter. J. Appl. Math. Mech. **22** (1958), 922–942 (657–672 Prikl. Mat. Meh.).

The author considers some classical problems for lth order partial differential equations in which the boundary data depend on an arbitrarily large parameter k. Illustrating his method in two dimensions, he considers the equation

$$L(\Phi) = \sum_{\nu=0}^{\nu=l} \sum_{j=0}^{j=\nu} a_{j,\nu-j}^{(\nu)} \frac{\partial^\nu \Phi}{\partial \alpha^j \partial \beta^{\nu-j}} = 0$$

in a simply connected region $\bar{\Gamma} = \Gamma + \gamma$ bounded by a contour γ. He takes α and β to be similar to polar coordinates with γ given by $\alpha = \alpha_0 > 0$, Γ determined by $0 \leqq \alpha \leqq \alpha_0$, $0 \leqq \beta \lesssim 2\pi$, and the usual ambiguity for $\alpha = 0$ and $\beta = 0, 2\pi$. The boundary conditions are

$$\frac{\partial^\mu \Phi}{\partial \alpha^\mu} = k^\mu g^{(\mu)} e^{ik\varphi} \quad (\mu = 0, 1, 2, \cdots, \tfrac{1}{2}l - 1),$$

where $g^{(\mu)}$ and φ are given functions of β which do not depend on k. The operator L is assumed not only to be elliptic, but also to have distinct complex characteristics near γ. It is then described how one may obtain a solution which is the sum of terms of the form

$$\Phi = e^{kf} \sum_{u=0}^{u=R} k^{-u} \Phi_u \quad (\Phi_0 \not\equiv 0),$$

where f and the $\Phi_0, \Phi_1, \cdots, \Phi_{R-1}$ are functions of α, β, and are independent of k. f is required to satisfy the characteristic equation of L, i.e.,

$$\sum_{j=0}^{j=l} a_{j,l-j}^{(l)} f_\alpha^{\,j} f_\beta^{\,l-j} = 0,$$

and to satisfy the boundary conditions $f = i\varphi(\beta)$, $\operatorname{Re} f_\alpha > 0$ on γ. The assumptions guarantee that there are $l/2$ distinct solutions in the neighborhood of γ. It is then shown how the $\Phi_0, \Phi_1, \cdots, \Phi_{R-1}$ may be determined (for each f) by solving the Cauchy problem for first order linear equations. The remainder, Φ_R, is a solution of the Cauchy problem for an lth order equation containing k. It is shown how Φ_R can be so determined as to remain bounded as $k \to \infty$ and such that the sum of the $l/2$ solutions satisfies the boundary conditions. An easy argument allows one to obtain solutions in the whole of Γ from those constructed in the neighborhood of γ.

The method is also applied to the Cauchy problem for completely hyperbolic equations of order l, and to the construction of particular integrals for some equations. Some examples and generalizations are given. The method described is related to that of Višik and Lyusternik [Uspehi Mat. Nauk **12** (1957), no. 5 (77), 3–122; MR **20** #2539] and Lax [Duke Math. J. **24** (1957), 627–646; MR **20** #4096].

M. Schechter (New York)

POTENTIAL THEORY

2803:

Müller, Reinhard. Die beiden ersten Randwertaufgaben der Potentialtheorie für einige oft vorkommende Kurven. Z. Angew. Math. Mech. **40** (1960), 278–281.

2804:

Mahovikov, V. I. Several problems for Laplace's equation. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 245–251. (Ukrainian. Russian and English summaries)

Author's summary: "Three solutions are presented of Laplace's equation. One solution enables us fully to satisfy the boundary conditions on a cylindrical surface and at several points on the ends of the cylinder. Applying the second together with the first solves the problem for a plate with given polyharmonics from the x, y functions on the end faces and with satisfaction on the cylindrical surface of the boundary conditions along several lines of $z = \text{const}$. The third solution makes it possible to reduce the space boundary problem for a body obtained on rotating the plane region D around the axis z to plane boundary problems on the boundary of region D."

2805:

Todorov, I.; Zidarov, D. Sur le caractère univoque de la solution du problème inverse dans la théorie du potentiel. Izv. Bŭlgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. 7 (1959), 283–293. (Bulgarian. Russian and French summaries)

Let T_1, T_2 be 3-dimensional bounded domains with smooth boundaries S_1, S_2. Let the boundary of $\overline{T_1 + T_2}$ be S^e and set $S^i = (S_1 + S_2) \cap (\operatorname{int} \overline{T_1 + T_2})$. Let 0 be a fixed point and let $\mathbf{R}$ be the vector from 0 to any point in the space. Further, let ν_0 be a fixed unit vector and ν the outwardly directed normal vector at points of S_1 and S_2. A mass of density 1 on T_i forms a Newtonian potential, denoted by φ_i. We finally denote by $(\mathbf{A}, \nu)$ the scalar product of $\mathbf{A}$ and ν, and assume that both $S^e \cap S_1$ and $S^e \cap S_2$ are non-empty. Theorem 1: Let (1) $\int_{S^i} |(\mathbf{R}, \nu)| ds \leqq \int_{S^e} |(\mathbf{R}, \nu)| ds$. If $\varphi_1 = \varphi_2$ outside $T_1 + T_2$, then $T_1 = T_2$. Theorem 2: Let (2) $\int_{S^i} |(\nu_0, \nu)| ds \leqq \int_{S^e} |(\nu_0, \nu)| ds$. If $\varphi_1 = \varphi_2$ outside $T_1 + T_2$, then $T_1 = T_2$. Theorem 1 was proved, for special domains T_1, T_2, by P. S. Novikov [Dokl. Akad. Nauk SSSR **18** (1938), 165–168] and L. N. Sretenskiĭ [ibid. **99** (1954), 21–22; MR **16**, 699]. For two dimensions, Yu. A. Šaškin [ibid. **115** (1957), 64–66; MR **20** #4711a] obtained theorems of a similar nature. The proof (of both theorems) is reduced to showing that if $T_1 \neq T_2$ then (3) $\int_{T_1} u\,dV - \int_{T_2} u\,dV = 0$ cannot hold for any harmonic function u in $\overline{T_1 + T_2}$. (3) is next transformed into (4)

$\int_{S_1} \bar{u}ds - \int_{S_2} \bar{u}ds = 0$ for any harmonic function $\bar{u}$ in $\overline{T_1+T_2}$. The author finally constructs a sequence $\bar{u}=\bar{u}_n$ of harmonic functions, and using (4) he derives a contradiction to (1) (and similarly for (2)). {Reviewer's remark: The author's last step is based on inequality (9) which is not justified; $\leq$ is valid, and therefore he establishes theorems 1, 2 only provided (1), (2), hold with $<$. The gap however can be closed by modifying his construction of G_n.} The inequalities in (1), (2) are optimal.

A. Friedman (Minneapolis, Minn.)

2806:

Godefroid, M. Une propriété des fonctions B.L.D. dans un espace de Green. Ann. Inst. Fourier. Grenoble **9** (1959), 301–304.

Let $\mathcal{E}$ be a Green space (natural domain for harmonic functions). Let G_P be the Green function of the space, with pole P. The orthogonal trajectories of the level curves $G_P=$ const are the "Green lines from P", and the measure of a set of these lines is the measure of the solid angle determined by their initial directions at P, normalized to have maximum value 1. Let u be a Beppo-Levi-Deny function on $\mathcal{E}$, that is, a function which is the limit on $\mathcal{E}$ both pointwise (neglecting a set of capacity 0) and in Dirichlet norm of a sequence of functions with continuous square-summable gradients. Let u_l^λ be the value of u at the point where the Green line l from P meets the boundary of the set where $G_P > \lambda$. Brelot [same Ann. **5** (1953/54), 317–419; MR **17**, 603] has proved that $\lim_{\lambda\to 0} u_l^\lambda$ exists as an L_1 limit, and the author now proves that the limit also exists almost everywhere, that is, for almost every l.

J. L. Doob (Urbana, Ill.)

2807:

Pogorzelski, W. Propriétés des dérivées tangentielles d'une intégrale de l'équation elliptique. Ann. Polon. Math. **7** (1960), 321–339.

This paper is concerned with the properties of a generalisation of the potential of a surface distribution of matter attracting according to the Newtonian law. This generalised potential is of the form

$$U(A) = \iint_S \Gamma(A, Q)\varphi(Q)dQ,$$

where $\varphi(Q)$ is the "density", Γ is the fundamental solution of a partial differential equation of elliptic type in n independent variables and S is a closed surface.

If P is a point of S, and if $V(A)$ is the value at a point A within S of the derivative of $U(A)$ in a direction parallel to a tangent to S at P, conditions are found under which $V(A)$ tends uniformly to a limit as $A\to P$ in any manner, and certain properties of the limit function are obtained.

E. T. Copson (St. Andrews)

2808:

Vašarin, A. A. Boundary properties of functions of class $W_2^1(\alpha)$ and their application to a boundary problem of mathematical physics. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 421–454. (Russian)

The author studies boundary function g for functions f, on a bounded domain Ω in n space, which belong to the class $W_2^1(\alpha)$; this class was one of the classes $W_2^k(\alpha)$ defined by the author in Dokl. Akad. Nauk SSSR **117** (1957), 742–744 [MR **20** #1113]. He proves an existence and uniqueness theorem for a precisely formulated boundary problem.

M. M. Day (Urbana, Ill.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 2901.

2809:

Naftalevič, A. G. A system of two difference equations. Mat. Sb. (N.S.) **47 (89)** (1959), 131–140. (Russian)

In the first three sections of this paper the author considers a system

$$(1)\qquad \begin{aligned} f(z+n\alpha) &= \sum_{k=0}^{n-1} p_k(z)f(z+k\alpha)+a(z),\\ f(z+m\beta) &= \sum_{j=0}^{m-1} q_j(z)f(z+j\beta)+b(z), \end{aligned}$$

where $p_0(z)\neq 0$, $q_0(z)\neq 0$ and $\operatorname{Im}(\alpha/\beta)\neq 0$, with initial data given on the mesh points of the parallelogram with vertices z, $z+(n-1)\alpha$, $z+(n-1)\alpha+(m-1)\beta$, and $z+(m-1)\beta$. Necessary and sufficient compatibility conditions are derived which guarantee that the system (1) determines a unique value of $f(z)$ at each mesh point of the complex z-plane.

In section 4 a special system of type (1), namely

$$(2)\qquad \begin{aligned} f(z+\alpha) &= p(z)f(z)+a(z),\\ f(z+\beta) &= q(z)f(z)+b(z), \end{aligned}$$

is treated with the compatibility conditions satisfied; $p(z)$, $q(z)$, $a(z)$ and $b(z)$ given meromorphic functions of z and z free to range throughout the complex plane. Meromorphic solutions of system (2) are constructed.

Section 5 is devoted to the determination of meromorphic solutions of the equation

$$P(z)f(z+\alpha)+Q(z)f(z+\beta) = R(z)f(z),\quad \operatorname{Im}(\alpha/\beta)\neq 0,$$

where $P(z)$, $Q(z)$ and $R(z)$ are given meromorphic functions, not identically zero, satisfying the condition

$$P(z)Q(z+\alpha)R(z+\beta) \equiv Q(z)R(z+\alpha)P(z+\beta).$$

The problem is reduced to a study of the solutions $E_\mu(z)$ of systems of the type

$$\begin{aligned} F(z+\alpha) &= \mu F(z),\\ F(z+\beta) &= (1-\mu)F(z), \end{aligned}$$

where μ takes on judiciously chosen constant values μ_j. Here sums and limits of sums of the type $\sum_{j=1}^{\nu} E_{\mu_j}(z)$ play an important role in the analysis.

In the final section 6 a nonhomogeneous difference equation

$$f(z+\alpha)+f(z+\beta) = f(z)+H(z),\quad \operatorname{Im}(\alpha/\beta)\neq 0,$$

where $H(z)$ is a given meromorphic function, is under consideration, and a method for constructing meromorphic solutions is outlined.

H. L. Turrittin (Minneapolis, Minn.)

2810:

Vincze, Endre. Über die Charakterisierung der assoziativen Funktionen von mehreren Veränderlichen. Publ. Math. Debrecen **6** (1959), 241–253.

The theorem [J. Aczél, Bull. Soc. Math. France **76** (1948), 59–64; MR **10**, 685] that any continuous strictly monotonic binary associative function F is of the form

$$F(x, y) = \phi^{-1}(\phi(x)+\phi(y)),$$

where ϕ and its inverse ϕ^{-1} are continuous and strictly monotonic, is generalized to $n$ary functions. Different formulae are obtained according as n is odd or even. The author then considers the equation $f^m(x)=x$; and the question of whether the $n$ary function under investigation can be built up out of binary functions.

H. A. Thurston (Vancouver, B.C.)

2811:

Ganapathy Iyer, V. On a functional equation. II. Indian J. Math. **2**, 1–7 (1960).

[For part I see J. Indian Math. Soc. (N.S.) **22** (1956), 283–290; MR **19**, 642; cf. also ibid. **10** (1946), 17–28; MR **8**, 509.] The main assertion of this paper is that the number of polynomials $P_n(z)$ of degree n such that the functional equation $f(z^2)=P_n(z)f(z)$ have a polynomial solution $f(z)$ is equal to 2^n. Some general remarks are also made about the functional equations $f(z^m)=P_n(z)f(z)$ ($P_n(z)$, $f(z)$ being again polynomials) and $f[g(z)]=h(z)f(z)$ ($f(z)$, $g(z)$, $h(z)$ being integral functions).

J. Aczél (Debrecen)

SEQUENCES, SERIES, SUMMABILITY

2812:

Jasek, B. Über Umordnung von Reihen. Colloq. Math. **7** (1959/60), 257–259.

The following theorem is proved. If $\sum_{n=1}^{\infty} a_n=A$ is a convergent series with $\lim_{n\to\infty} na_n=0$, and if, after rearrangement in the order of its terms, a new series $\sum_{n=1}^{\infty} a_{N_n}$ with $\lim_{n\to\infty} na_{N_n}=0$ is obtained, then $\sum_{n=1}^{\infty} a_{N_n}=A$. The condition $\lim_{n\to\infty} na_n=0$ is sufficient but not necessary. The author raises two questions: (1) Let $na_n\to 0$ and $na_{N_n}\to 0$ and let the series $\sum_{n=1}^{\infty} a_n$ be divergent. Is it true that $\lim_{n\to\infty} (a_1+\cdots+a_n)/(a_{N_1}+\cdots+a_{N_n})=1$? (2) How does one characterize the class of all rearrangements (permutations of subscripts) which do not change the limit of the series with $na_n\to 0$?

J. W. Andrushkiw (Newark, N.J.)

2813:

Vernotte, Pierre. La sommation des séries divergentes à termes positifs: difficultés introduites par leur valeur complexe. C. R. Acad. Sci. Paris **250** (1960), 1785–1786.

Présentation d'un procédé heuristique permettant d'attacher une somme complexe à certaines séries divergentes à termes réels positifs. *J. Kuntzmann* (Grenoble)

2814:

Matsuoka, Yoshio. A note on theorems of Sokolin. Sci. Rep. Kagoshima Univ. No. 8 (1959), 15–23.

A. S. Sokolin [Uspehi Mat. Nauk **13** (1958), no. 1 (79), 193–200; MR **20** #1868] introduced some methods of summability, and discussed their regularity, products, and inclusion relations. He asserted theorems 8–13 (p. 199) without proof; these theorems were concerned with relations between his methods of summability and the circular method (y, k) and the Meyer-König method (MK, r) [see G. H. Hardy, *Divergent series*, Clarendon, Oxford, 1949; MR **11**, 25; pp. 218–219; K. Zeller, *Theorie der Limitierungsverfahren*, Springer, Berlin, 1958; MR **20** #4119; pp. 140–145; and W. Meyer-König, Math. Z. **52** (1949), 257–304; MR **11**, 242]. The purpose of the present note is to give proofs of these theorems.

R. G. Cooke (London)

2815:

Borwein, D. On Borel-type methods of summability. Mathematika **5** (1958), 128–133.

Ist $s_n=\sum_0^n a_\nu$,

$$a(x) = \sum_0^\infty \frac{a_n x^{\alpha n}}{\Gamma(\alpha n+1)}, \quad s(x) = \sum_0^\infty \frac{s_n x^{\alpha n}}{\Gamma(\alpha n+1)} \quad (\alpha > 0),$$

so sei $s_n\to l\ (B', \alpha)$ falls $\int_0^\infty e^{-t}a(t)dt=l$, und $s_n\to l\ (\bar{B}, \alpha)$ falls $\alpha e^{-x}s(x)\to l\ (x\to\infty)$. In Verallgemeinerung des Falles $\alpha=1$ zeigt der Verf., dass $s_n\to l\ (\bar{B}, \alpha)$ genau dann gilt, wenn $s_n\to l\ (B', \alpha)$ und $a_n\to 0\ (\bar{B}, \alpha)$.

A. Peyerimhoff (Marburg)

2816:

Dorff, E. K.; Wilansky, A. Remarks on summability. J. London Math. Soc. **35** (1960), 234–236.

This paper contains seven independent notes on summability. Among others it is shown that there exists a regular triangular matrix A with $a_{nn}>0$, which is equivalent to convergence but is such that $A+tI$ is not equivalent to convergence for certain $t>0$. *P. Erdős* (Haifa)

2817:

Brudno, A. L. Topology of Toeplitz fields. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 771–780. (Russian)

Let R be the vector space of all bounded real sequences; let (a_k^n) be a Toeplitz matrix; let A be the linear transformation from R into R determined by (a_k^n); let $\mathfrak{A}$ be the convergence field of A, i.e., the set of sequences in R such that $A(x)$ is a convergent sequence, and let $\mathfrak{A}_0$ be the set of x in R for which $Ax=0$. If $x=\{x^1, \cdots, x^k, \cdots\}$, then $|x|$ and $\|x\|$ mean $\sup_k |x^k|$ and $\limsup_k |x^k|$.

For each positive function $\varepsilon(\nu)$, ν a positive number, define $\mathfrak{E}_{A\varepsilon}$ to be the convex hull of the $\bigcup_{\nu>0} V_\nu$, when $V_\nu=\{x: \|x\|<\nu \text{ and } \|Ax\|<\varepsilon(\nu)\}$. These sets $\mathfrak{E}_A$ are taken to be the neighborhoods of zero in the A-topology of R and also of $R/\mathfrak{A}_0$.

The fundamental theorem is that for two matrices a_k^n and b_k^n the first four of the following conditions are equivalent and imply the fifth. (1) $\mathfrak{B}\supseteq\mathfrak{A}$. (2) The B-topology is coarser than the A-topology; that is, each $\mathfrak{E}_B$ contains an $\mathfrak{E}_A$. (3) There is an $\mathfrak{E}_A$ contained in some $\mathfrak{E}_B$. (4) If the A-topology is used on the domain and the $\|\ \|$ in the range, B is continuous. (5) If $\mathfrak{B}(x)=\lim_n B(x)$ for each x in $\mathfrak{B}$, then $\mathfrak{B}(x)$ is continuous in the A-topology of $\mathfrak{B}$.

This implies the known result that if $\mathfrak{A}=\mathfrak{B}$ then $\mathfrak{A}(x)=\mathfrak{B}(x)$ for all x in $\mathfrak{A}$. *M. M. Day* (Urbana, Ill.)

2818:

Boyd, A. V. n-ary transformations of sequences. Proc. Edinburgh Math. Soc. **11** (1958/59), 221–222.

A transformation $T(a_1, a_2, \cdots, a_{n-1})$ which takes the sequence $\{s_i\}$ $(i=0, 1, 2, \cdots)$ into the sequence $\{s_i'\}$, where $s_i'=\sum_{r=1}^{n} a_r s_{i+1-r}$ $(i=0, 1, 2, \cdots)$, with $s_m=0$ when m is a negative integer, and where $a_1, a_2, \cdots, a_n$ are real numbers with sum unity, is called an n-ary transformation. The author generalizes the result obtained for ternary transformation [Borwein and Boyd, Proc. Edinburgh Math. Soc. **11** (1958/59), 175–181; MR **21** #7381] and shows that the n-ary transformation is equivalent to convergence if and only if all the zeros of $f(x)=\sum_{r=1}^{n} a_r x^{n-r}$ lie in the region $|x|<1$. A method is given to determine the set of points $(a_1, a_2, \cdots, a_{n-1})$ in the $(n-1)$-space.

J. W. Andrushkiw (Newark, N.J.)

2819:

Petersen, G. M. Almost convergence and the Buck-Pollard property. Proc. Amer. Math. Soc. **11** (1960), 469–477.

The author discusses L-regular methods of summation $A=(a_{mn})$, $m, n=1, 2, \cdots$, which map the set of all almost convergent sequences in itself, and absolutely regular methods, which map it into the set of convergent sequences. His main results are the following. (1) A is L-regular if and only if

$$\frac{1}{k}\sup_m \sum_{n=1}^{\infty} |(a_{mn}+\cdots+a_{m+k-1,n}) - (a_{m,n+1}+\cdots+a_{m+k-1,n+1})| \to 0$$

as $k\to\infty$. (2) Let A be absolutely regular. There exists a method B, whose matrix consists of lines b_n, $n=1, 2, \cdots$, of the type $b_n=m^{-1}$, $p+1\leqq n\leqq p+m$, $p, m=1, 2, \cdots$, and contains for each given m only a finite number of these lines, such that A sums all bounded sequences summed by B.

G. G. Lorentz (Syracuse, N.Y.)

2820:

Agnew, Ralph Palmer. Partial sums and transforms of Tauberian series. Ann. of Math. (2) **71** (1960), 395–407.

The question treated is the existence and the smallest possible value of the constant A such that

$$(*)\quad \limsup_{\alpha\to\infty} \Big|\sum_{k=1}^{\infty} \varphi(\lambda_k t)u_k - \sum_{\lambda_k\leqq T} u_k\Big| \leqq A \limsup_{n\to\infty} \frac{\lambda_n|u_n|}{\lambda_n-\lambda_{n-1}}.$$

Here $0<\lambda_1<\lambda_2<\cdots$, $\lambda_n\to\infty$, $t=t(\alpha)\to 0$, $T=T(\alpha)\to\infty$ for $\alpha\to\infty$, and $t(\alpha)T(\alpha)$ remains between two positive bounds. The inequality (*) is to hold for all series $\sum u_k$ with the right member of (*) finite. The sufficient conditions for the existence of an A are: $\varphi(t)$ is integrable over each interval (α, β), $\alpha>0$, $\beta<\infty$; and $\int_0^a t^{-1}\psi_1(t)dt<+\infty$, $\int_b^\infty t^{-1}\psi_2(t)dt<+\infty$ for some positive a, b, with $\psi_1(t)=\sup_{u\leqq t}|1-\varphi(u)|$, $\psi_2(t)=\sup_{u\geqq t}|\varphi(u)|$. In case $\lambda_{n+1}/\lambda_n\to 1$, the smallest possible value of A is determined. This improves some results of Delange [Ann. Sci. École Norm. Sup. (3) **67** (1950), 99–160; MR **12**, 253] and Rajagopal [Math. Z. **60** (1954), 142–147; Proc. Indian Acad. Sci. Sect. A **39** (1954), 272–281; MR **16**, 124, 125].

G. G. Lorentz (Syracuse, N.Y.)

APPROXIMATIONS AND EXPANSIONS

See also 2909, B3692.

2821:

Saxena, R. B. Convergence of interpolatory polynomials, (0, 1, 2, 4) interpolation. Trans. Amer. Math. Soc. **95** (1960), 361–385.

The author continues his study of (0, 1, 2, 4) interpolation, i.e., the interpolation which concerns n given points in $[-1, 1]$ when the values of the function, its first, second, and fourth derivatives are prescribed at these n points [cf. Saxena and A. Sharma, Acta Math. Acad. Sci. Hungar. **9** (1958), 345–358; **10** (1959), 157–175; MR **21** #2138, 3699a; and Saxena, Ph.D. Thesis, Lucknow University, Ch. 2, pp. 24–50]. Here is considered the convergence of the polynomials of interpolation [cf. last reference above]

$$\text{(A)}\quad R_n(x) = \sum_{v=1}^{n} a_v A_v(x) + \sum_{v=1}^{n} b_v B_v(x) + \sum_{v=1}^{n} c_v C_v(x) + \sum_{v=1}^{n} d_v D_v(x),$$

where $A_v(x)$, $B_v(x)$, $C_v(x)$, $D_v(x)$ are polynomials each of degree $\leqq 4n-1$. Let $1=x_{1n}>x_{2n}>\cdots>x_{n-1,n}>x_{nn}=-1$, $n=4, 6, \cdots, 2k, \cdots$, where the x_{vn} are the zeros of $\pi_n(x)=n(n-1)\int_{-1}^{x} P_{n-1}(t)dt=(1-x^2)P'_{n-1}(x)$, where $P_k(x)$ is the kth Legendre polynomial such that $P_k(1)=1$. Form the interpolatory polynomials (cf. (A) above) $A_{vn}(x)$, $B_{vn}(x)$, $C_{vn}(x)$, $D_{vn}(x)$ for each $n=2k$. Let $f(x)$ be defined in $[-1, 1]$. Consider the sequence of polynomials

$$\text{(B)}\quad R_n(x, f) = \sum_{v=1}^{n} f(x_{vn})A_{vn}(x) + \sum_{v=1}^{n} f'(x_{vn})B_{vn}(x) + \sum_{v=1}^{n} c_{vn}C_{vn}(x) + \sum_{v=1}^{n} d_{vn}D_{vn}(x)$$

with arbitrary numbers c_{vn} and d_{vn}. Then the following theorem is proved. Let $f(x)$ have continuous derivative of order 2 in $[-1, 1]$ with continuity modulus $\omega(\delta)$ of $f''(x)$ such that $\int_0 [\omega(t)/t^{3/2}]dt$ exists. Suppose that for arbitrarily small $\varepsilon>0$, for $n>n_0(\varepsilon)$ and $v=1, 2, \cdots, n$, $|c_{vn}|\leqq \varepsilon n$, $|d_{vn}|\leqq \varepsilon n^3$; then the sequence $R_n(x, f)$ converges to $f(x)$ uniformly in $[-1, 1]$.

E. Frank (Chicago, Ill.)

2822:

Popov, A. A. Orthogonal polynomials of arbitrary degree. Inžen. Sb. **26** (1958), 280–284. (Russian)

The author obtains the representation of a function $f(x)$ in the form $Ax^a+Bx^b+Cx^c+Dx^d+\cdots$. The coefficients $A, B, C, \cdots$ are determined from the conditions of minimization of the approximation in the mean over the interval $(0, l)$. For $a=0$, $b=1$, $c=2$, $\cdots$ this form becomes the Legendre polynomial representation of $f(x)$. Two other examples are given in which the exponents are the consecutive odd integers and the reciprocals of the natural numbers, respectively.

C. G. Maple (Ames, Iowa)

2823:

Meyer-König, W.; Zeller, K. Bernsteinsche Potenzreihen. Studia Math. **19** (1960), 89–94.

The authors call the series

$$P_n(x) = (1-x)^n \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} f\left(\frac{k}{k+n}\right) x^k$$

"the nth Bernstein power series" of the function $f(x)$, which is assumed to be defined for $0 \leq x < 1$. The $P_n(x)$ converge for all large n and satisfy $\lim_{n\to\infty} P_n(x) = f(x)$, uniformly for $x_1 \leq x \leq x_2$ $(0 \leq x_1 < x_2 < 1)$, for example if $f(x)$ is continuous in $[x_1, x_2]$ and if $|f(x)| \leq Me^{a/(1-x)}$, $0 \leq x < 1$. *G. G. Lorentz* (Syracuse, N.Y.)

2824:

Widder, D. V. Expansions in series of homogeneous polynomial solutions of the two-dimensional wave equation. Duke Math. J. **26** (1959), 591–598.

The convergence of expansions $u(x, y) = \sum P_n(x, y)$ is discussed, where P_n are homogeneous polynomials satisfying the wave equation. The series is shown to converge in a rectangle $|x+y| < \rho_1$, $|x-y| < \rho_2$ and possibly on portions of the extended diagonals. The Taylor expansion of such functions is also considered and convergence is shown for a square $|x| + |y| < \rho$ and again possibly on the extended diagonals. Reference is given to other papers in which the cases for P_n satisfying the harmonic equation and the heat equation are discussed.

J. Blackman (Syracuse, N.Y.)

2825:

Widder, D. V. Expansions in series of homogeneous polynomial solutions of the general two-dimensional linear partial differential equation of the second order with constant coefficients. Duke Math. J. **26** (1959), 599–603.

Expansions of the form $u(x, y) = \sum P_n(x, y)$ are considered where the P_n are homogeneous polynomials of degree n which satisfy the general linear second order partial differential equation with constant coefficients. The interesting equations can be brought to the form $u_{xx} + Nu_{yy} = 0$. The region of convergence can then be described as an ellipse, a parallelogram, and a strip in the elliptic, hyperbolic and parabolic cases respectively. In each case there may also be convergence on certain lines lying outside the indicated regions. The convergence of the double Taylor series expansion of $u(x, y)$ is also discussed. These theorems are generalizations of earlier results.

J. Blackman (Syracuse, N.Y.)

2826:

Price, J. J. Orthonormal sets with non-negative Dirichlet kernels. Trans. Amer. Math. Soc. **95** (1960), 256–262.

Haar [Math. Ann. **69** (1910), 331–371] defined a special system of orthonormal functions on [0, 1]. These have the property that the associated Dirichlet kernels $D_n(s, t) = \sum_{i=0}^{n-1} f_i(s) f_i(t)$ are always non-negative. This property is important in examining convergence for the expansion of a given function in terms of the orthonormal system. The author shows that the property that the Dirichlet kernels of an orthonormal system be all non-negative essentially implies that the system is a Haar system of a slightly more general type than that defined by Haar. The author gives appropriate definitions for a generalized Haar system on a measure space (S, μ) and proves his results in this context.

S. J. Taylor (Ithaca, N.Y.)

2827:

Baxter, Glen. Polynomials defined by a difference system. Bull. Amer. Math. Soc. **66** (1960), 187–190.

Let $f(t)$ be integrable on $[-\pi, \pi]$ with Fourier coefficients A_n. If the determinants $D_n = \det(A_{j-i})$ $(i, j = 0, 1, \cdots, n)$ are all non-zero, there is an essentially unique bi-orthogonal system of polynomials $\phi_n(z)$ of degree n in z, $\psi_n(z)$ of degree n in z^{-1}, such that

$$(*) \qquad (2\pi)^{-1} \int_{-\pi}^{\pi} \phi_n(z) \psi_m(z) f(t)\, dt = \delta_{mn} \quad (z = e^{it}).$$

The results announced state that such biorthogonal systems, under suitable regularity conditions, are equivalent to the solutions of a system of difference equations: starting with two complex sequences α_n, β_n, the system

$$u_n(z) - u_{n-1}(z) = \alpha_n z^n v_n(z),$$
$$v_n(z) - v_{n-1}(z) = \beta_n z^{-n} u_n(z),$$

$n \geq 0$, u_0 and v_0 constant, yields $\phi_n(z) = z^n v_n(z) k_n^{-1}$, $\psi_n(z) = z^{-n} u_n(z) k_n^{-1}$ satisfying (*) with

$$f(t) = k_\infty^2 [\phi^+(e^{it}) \phi^-(e^{it})]^{-1}$$

where ϕ^+ $[\phi^-]$ is the limit of u_n $[v_n]$ on $|z| \leq 1$ $[|z| \geq 1]$. These results extend the theory of G. Szegő [U. Grenander and G. Szegő, *Toeplitz forms and their applications*, Univ. Calif. Press, Berkeley, Calif., 1958; MR **20** #1349], which covers only the case when f is non-negative, and provide methods of attack on analogous continuous problems which are of interest in probability theory.

F. L. Spitzer (Princeton, N.J.)

2828:

Tandori, Károly. Über die orthogonalen Funktionen. VIII. Eine notwendige Bedingung für die Konvergenz. Acta Sci. Math. Szeged **20** (1959), 245–251.

Let $\{a_n\}$ be a sequence of positive numbers with $\sum a_n^2 < \infty$. To each sequence $s = \{n(i)\}$ of integers with $0 = n(1) < n(2) < \cdots$, define

$$A_k^2(s) = a^2_{n(k)+1} + \cdots + a^2_{n(k+1)} \quad (k = 1, 2, \cdots).$$

Let I denote the class of sequences s such that $\{A_k(s)\}$ is non-increasing. The author shows that if for each orthonormal system $\{\varphi_n\}$ on $[a, b]$ the series $\sum a_n \varphi_n(x)$ converges almost everywhere in $[a, b]$ then $\sum A_k^2(s) \log^2 k < \infty$ for each $s \in I$. The above result contains the classical results of D. Menchoff [Fund. Math. **4** (1923), 82–105] and H. Rademacher [Math. Ann. **87** (1922), 112–138] as well as an earlier result of the author [same Acta **18** (1957), 57–130; MR **19**, 851]. A companion result is given for $(C, 1)$ summability. *P. Civin* (Gainesville, Fla.)

2829:

Lorentz, G. G. Approximation of smooth functions. Bull. Amer. Math. Soc. **66** (1960), 124–125.

Let $\omega(h)$ be a continuous subadditive function, $\omega(h) \nearrow$, $\omega(0) = 0$, $h \geq 0$. A denotes a compact metric space, C_1^ω the set of all real-valued functions f on A with $|f(x)| \leq 1$, $|f(x) - f(x')| \leq \omega(h)$, $h = \rho(x, x')$. If A is a q-dimensional cube, p a natural number, $0 \leq \alpha \leq 1$, $C_1^{p+\alpha}$ denotes the class of all functions with partial derivatives of order p in Lip α. Let $G = \{g_n\}$ be a sequence of continuous functions on A.

$$E_n(f) = \inf \left\| f - \sum_1^n a_i g_i \right\|; \quad \mathscr{E}_n(W) = \sup E_n(f), \quad f \in W.$$

The author states the following general principle: An estimate of $\mathscr{E}_n(C_1^{p+\alpha})$ from below can be given for an

arbitrary system G, and the trigonometric system is close to the best possible. (1) Let $\delta=\delta(n)$ be the largest number such that there exist n points of A with mutual distances $\geqq\delta$. Then for each G, $\mathcal{E}_n(C_1^{\omega})\geqq\frac{1}{2}\omega(\delta(n+1))$. (2) If A is a q-dimensional cube, then for some constant B and each G (for example in L^1 norm) $\mathcal{E}_n(C_1^{p+\alpha})\geqq Bn^{-(p+\alpha)/q}$ $(p=0, 1, \cdots; 0<\alpha<1)$. From these theorems are obtained similar results for $E_n(f)$. No proofs are given.

M. Tomić (Belgrade)

2830:

Kawata, Tatsuo. Some Fourier integral theorems. Kōdai Math. Sem. Rep. **11** (1959), 77–87.

Given the functions f and K in $(-\infty, \infty)$, we may, under suitable conditions, represent f by means of the kernel K, as follows:

$$\lim_{w\to\infty}\int_{-\infty}^{\infty} f\left(x+\frac{t}{w}\right)K(t)dt = f(x)\int_{-\infty}^{\infty}K(t)dt.$$

If, for instance, f, $K\in L_1(-\infty, \infty)$, then one is led to a general summability theorem for Fourier integrals. [See S. Bochner, *Lectures on Fourier integrals*, Princeton Univ. Press, Princeton, N.J., 1959; MR **21** #5851.] In this paper the author considers the case in which f, $K\in L_2(-\infty, \infty)$. Instead of a general limit-relation as above, he gets only an estimate of the order of magnitude of the integral

$$\int_{-\infty}^{\infty} f\left(x+\frac{t}{w}\right)K(t)dt$$

and of its two-dimensional analogue

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(x+\frac{s}{w}, y+\frac{t}{w}\right)K(s)\overline{K(t)}dsdt.$$

And then for a special choice of the function $f(x, y)$, he obtains a two-dimensional analogue of the limit-relation. He mentions that an equivalent result was proved by E. Parzen [Ann. Math. Statist. **28** (1957), 329–348; MR **19**, 587].

K. Chandrasekharan (Bombay)

2831:

Butzer, P. L. Representation and approximation of functions by general singular integrals. IA, IB. Nederl. Akad. Wetensch. Proc. Ser. A **63**=Indag. Math. **22** (1960), 1–24.

This paper contains a rather extensive survey of approximations to functions $f(x)\in L^1(-\infty, \infty)$ by integrals of the form $\tau_\rho(x)=\int_{-\infty}^{\infty} f(x-u)\rho K(\rho u)du$ as $\rho\to\infty$. Here $K(u)\geqq 0$, $-\infty<u<\infty$ and $\int_{-\infty}^{\infty} K(u)du=1$. By imposing various conditions on K a variety of results are obtained. We give one example. Let $K(u)$ be even and continuous for $u=0$. If the moments $m_2, \cdots, m_{2k+1}$, where $m_j=\int_{-\infty}^{\infty}|u|^jK(u)du$, are finite, and if $f^{(2k)}$ exists and is continuous on $(-\infty, \infty)$ with modulus of continuity $\omega(\delta)$, then

$$\tau_\rho(x) = f(x)+2\sum_{i=1}^{k}\frac{1}{(2i)!}\frac{m_{2i}}{\rho^{2i}}f^{(2i)}(x)+O[\rho^{-2k}\omega(1/\rho)].$$

Under the same assumptions the degree of approximation may be improved by the following ingenious scheme. Let $\mathfrak{T}_\rho^{[2]}=2\tau_{2^{1/2}\rho}(x)-\tau_\rho(x)$, $\mathfrak{T}_\rho^{[4]}(x)=2\mathfrak{T}_{2^{1/4}\rho}^{[2]}(x)-\mathfrak{T}_\rho^{[2]}(x)$, etc.; then $\mathfrak{T}_\rho^{[2k]}(x)=f(x)+O[\rho^{-2k}\omega(1/\rho)]$. By means of the Poisson summation formula these methods may also be applied to functions $f(x)$ on $(-\infty, \infty)$ which are periodic with period 2π. In this context the construction described above yields a very elegant proof of various classical approximation theorems, as well as some new results. The paper closes with a series of theorems which assert that if $\|f(x)-\tau_\rho(x)\|_\infty$ goes to 0 too rapidly as $\rho\to\infty$ then f must vanish identically.

I. I. Hirschman, Jr. (Huntington, W. Va.)

2832:

Freĭ, T. [Frey, Tamás]. On the asymptotic behavior of orthogonal sequences of polynomials. Mat. Sb. (N.S.) **49 (91)** (1959), 133–180. (Russian)

Le résultat principal de cet article peut être exprimé de la manière suivante: Soient $f(t)$, $g(t)\in L_{2\pi}\equiv L(0, 2\pi)$, $\log f$, $\log g\in L_{2\pi}$, $k(t)\equiv g(t)/f(t)$, $k(t_0)=1$. Désignons par $\{F_n(z)\}$ et $\{G_n(z)\}$ les polynômes orthogonaux sur le cercle $|z|=1$, par rapport aux fonctions f et g. Soit

$$D(z;f) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\log f(t)\,\frac{1+ze^{it}}{1-ze^{-it}}\,dt\right\};$$

$\overline{D(\bar{z};f)}=\bar{D}(z;f)$. Si $F_n(z_0)$, $z_0=e^{it_0}$ admet au point z_0 une représentation asymptotique de la forme

$$F_n(e^{it_0}) = e^{int_0}\{\bar{D}(e^{-it_0};f)\}^{-1}+o\{|\bar{D}(e^{-it_0};f)|^{-1}\},$$

et si les conditions suivantes soient remplies: (1) $0<k_1\leqq k(t)\leqq k_2$, $-\pi\leqq t\leqq\pi$; (2) $|k(t)-k(t_0)|\leqq C_1|t-t_0|q(|t-t_0|)$, $-\pi\leqq t-t_0\leqq\pi$, où $q(\tau)\uparrow$, $q(\tau)/\tau\downarrow$, $\int_0^{2\pi} q(\tau)/\tau d\tau<\infty$; (3) $|F_n(e^{it})\sqrt{f(t)}|<C_2$, $-\pi\leqq t-t_0\leqq\pi$; alors la suite $\{G_n(z)\}$ admet au point z_0 aussi une représentation asymptotique. L'auteur déduit ensuite une suite des théorèmes de ce genre. Ces résultats font une généralisation étendue des résultats classiques de Szegö [*Orthogonal polynomials*, Amer. Math. Soc., New York, 1939; MR **1**, 14; pp. 289–309] et spécialement de J. Korous [Rozpravy II České Akad. **48** (1938), no. 1]. La méthode de démonstration est aussi la généralisation des méthodes de Szegö et Korous. Les complications dans les démonstrations proviennent d'une part de la très générale forme de (2).

M. Tomić (Belgrade)

FOURIER ANALYSIS

See also 2830, 2910.

2833:

Gosselin, Richard P. Some theorems on L^p Fourier series. Trans. Amer. Math. Soc. **92** (1959), 291–301.

Let $f(x)\sim\sum a_ne^{inx}$ be the Fourier series of a function in L^p, $1<p\leqq 2$, with $S_n(x;f)$ denoting the nth partial sums. The author deals with the problem of proving

(*) $\qquad S_{n_k}(x;f)\to f(x)$ for a.e. x

for appropriate subsequences $\{n_k\}$. In a previous paper [Proc. Amer. Math. Soc. **7** (1956), 392–397; MR **18**, 303] the author obtained the following result. For every $f\in L^2$, there exists a subsequence $\{n_k\}$ of upper density one so that (*) holds. The subsequence depends on f but does not depend on x. Here the author obtains a generalization of this result for $f\in L^p$, which extends his previous partial results for L^p. Let $f\in L^p$, $1<p\leqq 2$; then there exists a subsequence $\{n_k\}$ with the following property: (1) if

$\sigma(n)$ denotes the number of terms of this subsequence not exceeding n, then

$$\limsup_{n\to\infty} \frac{(\log n)^{(2-p)/2(p-1)}\sigma(n)}{n} \geq 1;$$

(2) (*) holds. The proof of this theorem uses the Littlewood-Paley theory of Fourier series. The author also treats subsequences satisfying conditions of "lacunarity", as well as analogues of these results for Fourier integrals.
E. M. Stein (Chicago, Ill.)

2834:

Hsiang, Fu Cheng. On the Gibbs phenomenon for harmonic means. Proc. Amer. Math. Soc. **11** (1960), 284–290.

Let f be a function of bounded variation. The author shows that at any simple discontinuity of f, the Fourier series of f exhibits a Gibbs phenomenon with respect to harmonic means with the Gibbs ratio $(2/\pi)\int_0^\pi t^{-1}\sin t\,dt$.
P. Civin (Gainesville, Fla.)

2835:

Hsiang, Fu Cheng. On the absolute summability of a Fourier series and its conjugate series. Proc. Amer. Math. Soc. **11** (1960), 32–38.

Let $f(x)$ be a Lebesgue integrable function of period 2π. Let $w(\theta, t)=f(\theta+t)-f(\theta)$,

$$w_p(t) = \left((2\pi)^{-1}\int_{-\pi}^{\pi} |w(\theta, t)|^p d\theta\right)^{1/p},$$

$$\omega(\theta, t) = \int_0^t \{f(\theta+u)-f(\theta)\}du,$$

$$\Omega_p(t) = \left((2\pi)^{-1}\int_{-\pi}^{\pi} |\omega(\theta, t)|^p d\theta\right)^{1/p}.$$

The author proves the following theorem which is a refinement of a result of H. C. Chow [J. London Math. Soc. **30** (1955), 439–448; MR **17**, 32]. If $1\leq p\leq 2$, $\int_{-\pi}^{\pi} w_p(t)dt<\infty$, and $\int_{-\pi}^{\pi}\Omega_p(t)t^{-2}dt<\infty$, then the Fourier series of f and its conjugate series are summable $|C, \alpha|$ for $\alpha>1/p$.
K. Chandrasekharan (Bombay)

2836:

Pati, T. On the absolute Riesz summability of Fourier series, its conjugate series and their derived series. Proc. Nat. Inst. Sci. India. Part A **23** (1957), 354–369.

Der Verf. untersucht die $|R, \lambda, \kappa|$-Summierbarkeit von Fourierreihen für das Verfahren mit $\lambda_n=\exp(\log n)^{1+1/\alpha}$, $\alpha>0$. Ist $f(x)\in L(-\pi, +\pi)$ mit 2π periodisch und ist für ein ganzes $\alpha\geq 1$ $\log (k/t)t^{-\alpha}\int_0^t(t-u)^{\alpha-1}\phi(u)du\in V(0, \pi)$ (für ein $k>\pi$), $2\phi(t)=f(x+t)+f(x-t)$, so ist die Fourierreihe von $f(t)$ für $t=x$ $|R, \lambda, \alpha+\delta|$-summierbar für jedes $\delta>0$. Entsprechende Sätze gelten für die konjugierte Reihe und die abgeleiteten Reihen. [Für $\alpha+1$ statt $\alpha+\delta$ vgl. T. Pati, Trans. Amer. Math. Soc. **76** (1954), 351–374; MR 15, 952.]
A. Peyerimhoff (Marburg)

2837:

Pati, T.; Sinha, S. R. On the absolute summability factors of Fourier series. Indian J. Math. **1**, no. 1, 41–54 (1958).

Let $f(t)$ be an integrable function with period 2π. Let

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2f(x)\},$$

$$\Phi_\alpha(t) = (\Gamma(\alpha))^{-1}\int_0^t (t-u)^{\alpha-1}\phi(u)du,$$

and $\phi_\alpha(t)=\Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t)$. Let h be an integer ≥ 0, $\{\lambda_n\}$ be a sequence satisfying: (i) $\Delta^k\lambda_n\geq 0$, $k=0, 1, \cdots, h+1$; (ii) $\sum n^{-1}\lambda_n<\infty$; (iii) $\sum n^h\Delta^{h+1}\lambda_n<\infty$. Assume $\int_0^t|\phi_h(u)|du=O(t)$. Let $f(t)\sim\sum A_n(t)$ be the Fourier series of f. Then the authors prove that $\sum\lambda_nA_n(x)$ is $|C, h+1+\delta|$ summable, for every $\delta>0$. Related results dealing with absolute summability factors may be found in M. T. Cheng, *Summability factors of Fourier series*, Duke Math. J. **15** (1948), 17–27 [MR **9**, 580], and T. Pati, *The summability factors of infinite series*, Duke Math. J. **21** (1954), 271–283 [MR **15**, 950].
E. M. Stein (Chicago, Ill.)

2838:

Pati, T. On the absolute Cesàro summability of Fourier series of functions of Lebesgue class L^p and some related problems in the theory of Fourier constants. Ann. Mat. Pura Appl. (4) **47** (1959), 181–195.

Analysing a theorem of Tsuchikura [Tôhoku Math. J. (2) **5** (1953), 52–66; MR **15**, 417; theorem V] the author is led to the following conjecture.

Let the complex Fourier series of the function $f(t)$, for which $f(t)t^{\beta-1/p}[\log(\beta/|t|)]^\alpha\in L^p(-\pi, \pi)$, $\alpha>1/q$, $\beta>\pi$, $1<p\leq 2$, $1/p+1/q=1$, be $\sum_{-\infty}^{\infty}c_ne^{int}$. Then, for every $\delta>1/p$, $\sum_{-\infty}^{\infty}(|n|+1)^{-\delta}|c_n|<\infty$.

The analogous theorem for $f(z)=\sum_1^\infty c_nz^n$ is effectively proved.
V. Vučković (Belgrade)

2839:

Matuszewska, W.; Orlicz, W. On the local tests for convergence of Fourier series. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 549–556. (Russian summary, unbound insert)

Let K be a certain class of continuous functions in $[a, b]$. The purpose of this paper is to show that, if the local condition W defined below is additionally imposed on the functions from K, then there exist in K functions which fail to satisfy W in any neighborhood of every point belonging to a certain denumerable set. This furnishes, if a suitable K is chosen, a proof that convergence-tests for Fourier series of one certain type never give necessary conditions for convergence. Let W denote a certain set of systems $[x, \xi, \delta]$, $\delta>0$, where $x\in K$, $\xi\in[a, b]$ and W fulfills the following conditions: (I) if $[x, \xi, \delta]\in W$, then $[x, \xi, \delta_1]\in W$ for $\delta_1<\delta$; (II) if $[x_n, \xi_n, \delta]\in W$, $\xi_n\to\xi_0$, $x_n(t)\to x(t)$ uniformly in $[a, b]$, $x\in K$, then $[x, \xi_0, \delta]\in W$; (III) if for every $\xi\in(\alpha, \beta)\subset[a, b]$, $[x_n, \xi, \delta]\in W$, then there exists a subsequence x_{n_i} such that $x_{n_i}(t)$ has a limit (finite or not) almost everywhere in (α, β). Theorem 1: Let $\varphi_n(t)$ be continuous and $|\varphi_n(t)|\leq M$ in $[a, b]$ for $n=1, 2, \cdots$. Let further

$$\liminf_{m\to\infty}\liminf_{n\to\infty}\int_\alpha^\beta(\varphi_n(t)-\varphi_m(t))^2dt > 0$$

for arbitrary $(\alpha, \beta)\subset[a, b]$. Denote by K the class of those functions x_a which are of the form $x_a=\sum_{i=1}^\infty a_i\varphi_i(t)$, where $a=\{a_i\}\in H^1$, i.e., $\sum_{i=1}^\infty|a_i|<\infty$. Write further $A_a=\{\xi: \xi\in[a, b], [x_a, \xi, \delta]\notin W$ for any $\delta>0\}$. Under the above

assumptions there exists in H^1 a residual set H^* such that for every $a \in H^*$, A_a is residual in $[a, b]$.—Two other theorems are given varying the conditions for W, and they are applied to three theorems about local tests for convergence of Fourier series. Example (th. 6): Let h_n be a sequence decreasing to zero. Let $f(t)$ be a function continuous and non-constant in [0, 1] and with period 1, and let $\omega_n \to \infty$ and $x_a(t) = \sum_{i=1}^{\infty} a_i f(\omega_i t)$. There exists a residual set $H^* \subset H^1$ and a set $\xi_1, \xi_2, \cdots$ everywhere dense in $[a, b]$ such that we have for $a \in H^*$,

$$\limsup_{n\to\infty} \frac{|x_a(\xi_i+h_n)-x_a(\xi_i)|}{h_n} = \infty$$

with $i = 1, 2, \cdots$. *G. Goes* (Evanston, Ill.)

2840:

Jeśmanowicz, L. Application of the Nörlund summability to the theory of localization for single and double trigonometric series. I. Ann. Polon. Math. **6** (1959), 217–240.

The author considers Nörlund means determined by sequences $\{A_n\}$ satisfying the conditions: (i) $A_n > 0$, $n \geq n_0$; (ii) $A_n \geq A_{n+1}$, $n \geq n_0$; (iii) $\sum A_n = \infty$; (iv) $1-(A_{n-1}/A_n) = O(n^{-1})$. He obtains results on the formal multiplication such as the following: The formal product of the series $\sum a_p$ and $\sum b_p$ is Nörlund summable to 0 with respect to $\{A_n\}$ whenever $a_p = o(A_{|p|})$, $b_p = O(p^{-4})$, and $\sum b_p = 0$. As an application he obtains theorems on the localization of series such as the following: If $a_p = o(A_{|p|})$ and, for $m \geq 1$, the function

$$F(x) = a_0(x^m/m!) + u_1 x^{m-1} + \cdots + u_m + \sum a_p (ip)^{-m} e^{ipx}$$

vanishes on an interval $[a, b]$, then in any interval $[c, d]$ interior to $[a, b]$ the trigonometric series $\sum a_p e^{ipx}$ is uniformly Nörlund summable to 0 with respect to $\{A_n\}$.

Comparable results are obtained relative to Nörlund summability of order k with respect to $\{A_n\}$.

P. Civin (Gainesville, Fla.)

2841:

Favard, J. Sur la sommation des séries de Fourier des fonctions périodiques à variation bornée. Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 2, 155–161 (1959). (Bulgarian and Russian summaries)

The author states and illustrates the principle that a summability method will be successful for the Fourier series of every function of bounded variation if and only if it works for I:

$$I(t) = \sum_1^{\infty} (1/k) \sin kt, \quad 0 \leq t \leq 2\pi.$$

Let the a_k and the b_k be the Fourier coefficients of the normalized function x of bounded variation on $[0, 2\pi]$, and let

$$\sigma_n^\gamma x(t) = \tfrac{1}{2}a_0 + \sum_{k=1}^{n-1} \gamma_n^k (a_k \cos kt + b_k \sin kt).$$

Then $\sigma_n^\gamma x \to x$ if and only if $\sigma_n^\gamma I \to I$. Among the kinds of convergence considered: (i) convergence for $0 < t < 2\pi$, boundedly on $(0, 2\pi)$; (ii) uniform convergence on every closed interval of continuity, boundedly on $(0, 2\pi)$.

J. Korevaar (Madison, Wis.)

2842:

Goldberg, Richard R. Averages of Fourier coefficients. Pacific J. Math. **9** (1959), 695–699.

The author proves that if $\psi(x)$ is of bounded variation in the interval $0 \leq x \leq 1$, and (a_n) is the sequence of Fourier coefficients of an L^p function, then the sequence $(1/n)\sum_{m=1}^{n} \psi(m/n) a_m$ $(n = 1, 2, \cdots)$ is also. The case $\psi(x) = 1$ was proved by G. H. Hardy [Messenger of Math. **58** (1929), 50–52]. *S. Izumi* (Sapporo)

2843:

Lehmer, D. H. On a problem of Hardy and Littlewood. J. London Math. Soc. **34** (1959), 395–396; 485.

The author gives counter-examples to the following proposition: If $1 \geq a_1 \geq a_2 \geq \cdots$, with $a_n \geq 0$ for each n, and $c_1, c_2, \cdots$ is a sequence of real numbers, such that $|c_1|$, $|c_2|, \cdots$ is a rearrangement of the sequence $a_1, a_2, \cdots$, and

$$f^*(\theta) = 1 + a_1 \cos\theta + a_2 \cos 2\theta + \cdots,$$

$$f(\theta) = 1 + c_1 \cos\theta + c_2 \cos 2\theta + \cdots,$$

then

$$\int_0^{\pi} |f^*(\theta)|\,d\theta \leq \int_0^{\pi} |f(\theta)|\,d\theta.$$

The example, "discovered by a fairly elaborate programme" of high-speed computing, is fairly simple: $a_n = c_n = 0$ for $n > 4$, while $a_1 = a_2 = a_3 = \frac{3}{4}$, $a_4 = \frac{1}{2}$, $c_1 = -\frac{3}{4}$, $c_2 = c_4 = \frac{3}{4}$, $c_3 = -\frac{1}{2}$. Counter-examples are given for the case in which f and f^* have a constant term $\frac{1}{2}$. The question was first raised by Hardy and Littlewood [same J. **23** (1948), 163–168; MR **10**, 448], and the use of high-speed computing to get a counter-example was suggested by J. E. Littlewood.

K. Chandrasekharan (Bombay)

2844:

Dzyadyk, V. K. Best approximation on classes of periodic functions defined by kernels which are integrals of absolutely monotone functions. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 933–950. (Russian)

Apart from some minor points, this paper contains detailed proofs of results outlined by the author in his earlier paper in Dokl. Akad. Nauk SSSR **129** (1959), 19–22 [MR **22** #870]. *G. G. Lorentz* (Syracuse, N.Y.)

2845:

Walmsley, Charles; Grant, Arthur S. G. The Fourier transform of $[c^{2k}+(x-x^{-1})^{2k}]^{-1}$ arising from study of tuned circuit spectra. Math. Comput. **14** (1960), 193–195.

In existing tables, the Fourier transform of $1/[c^{2k} + (x-x^{-1})^{2k}]$ is given only for $k = 1$ and $0 < c < 2$. Since the need for this Fourier transform in other cases has arisen recently in connection with a study of band pass filters, the authors have evaluated the Fourier transform in question (by contour integration) for $c > 0$ and k a positive integer. The complete result is given in the paper.

A. Erdélyi (Pasadena, Calif.)

2846:

Ishikawa, Hiroshi; Koizumi, Sumiyuki. On some theorems of the Fourier transform. J. Fac. Sci. Hokkaido Univ. Ser. I **14**, 225–230 (1959).

Inequalities of the type obtained by Hardy and Littlewood [Math. Ann. **97** (1926), 159–209] relating a function and its Fourier transform are obtained. Use is made of an as yet unpublished extension by the second named author of an interpolation theorem of Zygmund [J. Math. Pures Appl. (9) **35** (1956), 223–248; MR **18**, 321].

P. Civin (Gainesville, Fla.)

2847:

Iwasaki, Koziro. Some characterizations of Fourier transforms. Proc. Japan Acad. **35** (1959), 423–426.

If $F(x)$ in $(-\infty, \infty)$ is even, and $\varphi(x)$ has bounded support, then, subject to further conditions, if for all φ

$$\sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(nt)\varphi(t)dt = \sum_{-\infty}^{\infty} \varphi(n)$$

then $F(x) = \cos 2\pi x$. If $F(x)$ is not even, but furthermore

$$\int_{-\infty}^{\infty} F(xt)\varphi * \psi(t)dt = \int_{-\infty}^{\infty} F(xt)\varphi(t)dt \cdot \int_{-\infty}^{\infty} F(xt)\psi(t)dt$$

for any functions φ, ψ of bounded support and their convolution $\varphi * \psi$, then $F(x) = e^{2\pi ix}$ or $F(x) = e^{-2\pi ix}$.

S. Bochner (Princeton, N.J.)

2848:

Mamaĭ, L. V. On the theory of characteristic functions. Vestnik Leningrad. Univ. **15** (1960), no. 1, 85–99. (Russian. English summary)

Generalizing a result of Linnik [Dokl. Akad. Nauk SSSR **116** (1957), 735–737; MR **20** #331] it is proved that if we have $\prod_{j=1}^{\infty} f_j^{\alpha_j}(t) = \phi(t)$ in a real interval containing the origin, where the f_j's are characteristic functions and ϕ is regular in the strip $|\mathrm{Im}\,(t)| < M$ and nonvanishing there, and the sequence α_j is bounded away from zero, then the f_j's are regular in the strip and the above relation holds in the strip. Another theorem requires the relation to hold only for a sequence of t converging to zero.

K. L. Chung (Syracuse, N.Y.)

2849:

Gel'fand, I. M.; Sya, Do-šin [Shah, Tao-Shing]. On positive definite distributions. Uspehi Mat. Nauk **15** (1960), no. 1 (91), 185–190. (Russian)

Let K denote the vector space of all C^∞ functions on $(-\infty, \infty)$ with compact support, J the subspace of all even functions $\in K$. The authors call an even linear functional T on K positive definite on J if and only if $(T, \varphi * \bar{\varphi}) \geq 0$ for all $\varphi \in J$. When T is an ordinary even continuous function f this condition says that the kernel $\{f(x+t)+f(x-t)\}$ $(0<x<\infty, 0<t<\infty)$ is positive definite in the ordinary sense. For functions f of this kind Kreĭn has proved a representation theorem [Dokl. Akad. Nauk SSSR **53** (1946), 3–6; MR **8**, 277]. The authors extend this representation to all even linear functionals T on K which are positive definite on J (continuity is not required). They show that, symbolically,

$$T(x) = \int_0^\infty \cos x\lambda \, d\mu(\lambda) + \int_0^\infty \cosh x\lambda \, d\sigma(\lambda).$$

Here $\mu(\lambda)$ and $\sigma(\lambda)$ are (non-unique) positive measures. It is further shown that the representation remains valid and becomes unique if K is replaced by Z_2^2, that is, the vector space of all entire functions φ of order ≤ 2 such that $|\varphi(x)| \leq a \exp(-bx^2)$ for some $a, b > 0$.

J. Korevaar (Madison, Wis.)

2850:

Vasilach, Serge. Sur quelques extensions d'un théorème de Titchmarsh. Boll. Un. Mat. Ital. (3) **14** (1959), 386–398. (English summary)

The Titchmarsh theorem [Proc. London Math. Soc. (2) **25** (1926), 283–302; see pp. 287–288] states that if $f(t)$ is in L^1 on $[a, b]$ and $\int_a^b e^{zt}f(t)dt \equiv 0$ for all complex z, then $f(t) = 0$ a.e. on $[a, b]$. This is generalized to: if Γ_n is an open n-dimensional rectangle of R_n, μ is a measure on Γ_n and $\int_{\Gamma_n} e^{\zeta t} d\mu(t) \equiv 0$ in $\zeta = (z_1, \cdots, z_n)$, z_i complex with $\zeta t = \sum_1^n z_i t_i$, then $\mu \equiv 0$ on Γ_n. The extension to a real linear locally convex separated quasi-complete space F in whose dual F' there exists a denumerable set dense in the weak topology $\sigma(F; F')$ reads: if $f(t)$ on Γ_n to F is weakly integrable relative to the positive measure function μ on Γ_n and such that $\int_{\Gamma_n} e^{\zeta t}f(t)d\mu(t) = 0$ for all complex $\zeta = (z_1, \cdots, z_n)$, then $f(t) = 0$ on the closure $\bar{\Gamma}_n$ a.e. relative to μ.

T. H. Hildebrandt (Providence, R.I.)

2851:

Williamson, J. H. A theorem on algebras of measures on topological groups. Proc. Edinburgh Math. Soc. **11** (1958/59), 195–206.

In a well-known paper [Duke Math. J. **4** (1938), 420–436] Wiener and Pitt gave a construction of a complex function μ of finite variation on $]-\infty, \infty[$ such that $|\hat{\mu}(t)| = |\int_{-\infty}^{\infty} \exp(itx)d\mu(x)|$ is bounded away from 0 for all real t but $1/\hat{\mu}$ is not a Fourier-Stieltjes transform. The complexities of their construction have puzzled many readers, and the existence of a μ of the sort described was established in a quite different way by Šreĭder [Mat. Sb. (N.S.) **27** (**69**) (1950), 297–318; MR **12**, 420]. In group-theoretic terms, an intimately related problem is the following. Given a locally compact Abelian group G with measure algebra $\mathscr{M}(G)$, and adjoint defined by $\tilde{\lambda}(E) = \overline{\lambda(E^{-1})}$, is there a multiplicative linear functional τ on $\mathscr{M}(G)$ and a $\lambda \in \mathscr{M}(G)$ such that $\tau(\tilde{\lambda}) \neq \overline{\tau(\lambda)}$? Šreĭder answered this question affirmatively for $G =]-\infty, \infty[$, and his result was extended by Hewitt [Michigan Math. J. **5** (1958), 149–158; MR **21** #4992] to all G containing arbitrarily small elements of infinite order. In the present paper, the result is proved for all nondiscrete locally compact Abelian groups. The argument harks back to the original construction of Wiener and Pitt. The main theorem is: $\mathscr{M}(G)$ contains a measure λ such that $\lambda = \tilde{\lambda}$ and $\|\sum_{r=0}^n a_r\lambda^r\| = \sum_{r=0}^n |a_r|$ for all nonnegative integers n and complex numbers $a_0, \cdots, a_n$. This at once gives the asymmetry of $\mathscr{M}(G)$ and the Wiener-Pitt phenomenon. Hewitt and Kakutani have recently constructed even more extreme examples of this type [Illinois J. Math. **4** (1960), 553–574].

E. Hewitt (Seattle, Wash.)

2852:

Goldberg, Richard R. Watson transforms on groups. Ann. of Math. (2) **71** (1960), 522–528.

A comprehensive and purely operatorial characterization of a general class of integral transforms in $L_2(0, \infty)$ originally studied by G. N. Watson [Proc. London Math. Soc. (2) **35** (1933), 156–199] was given by S. Bochner [Ann. of Math. (2) **35** (1934), 111–115]. The basic result is as follows. Given $k(x)/x \in L_2(0, \infty)$ with the property $\int_0^\infty k(ay)\overline{k(by)}/y^2\,dy = \min(a, b)$, $a, b > 0$, there exists a

unitary transformation T in $L_2(0, \infty)$, called the Watson transform, such that if $Tf=g$, then

$$\int_0^x g(x)\,dx = \int_0^\infty \frac{k(xy)}{y} f(y)\,dy,$$

$$\int_0^x f(x)\,dx = \int_0^\infty \frac{k(xy)}{y} g(y)\,dy.$$

It was later shown by Bochner and the reviewer [*Fourier transforms*, Princeton Univ. Press, Princeton, N.J., 1949; MR 11, 173; p. 163] that to every unitary transformation in $L_2(0, \infty)$ representable by the Watson transform there corresponds a unitary transformation $\phi \to \psi$ of the form U: $\psi(\alpha) = T(\alpha)\phi(-\alpha)$ and U^{-1}: $\phi(\alpha) = \overline{T(-\alpha)}\psi(-\alpha)$ where $\phi, \psi \in L_2(-\infty, \infty)$, and $|T(\alpha)|^2 = 1$. Conversely every such transformation U derives from a certain Watson transform. It follows that the set of Watson transforms is in one-one correspondence with the set of functions of absolute value one almost everywhere. The author extends this result from $L_2(0, \infty)$ to $L_2(G)$ where G is a locally compact abelian group. The analysis goes through if one rephrases the above result as the author does in terms of the invariant measure for the positive real numbers considered as a multiplicative group, instead of the Lebesgue measure, and then effects an appropriate generalization. *K. Chandrasekharan* (Bombay)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 2885, B3628.

2853:

Pistoia, Angelo. Trasformate di funzioni di ripartizione. Boll. Un. Mat. Ital. (3) **14** (1959), 537–542. (English summary)

Let R be the class of distribution functions and C the class of characteristic functions of distribution functions [cf. H. Cramér, *Mathematical methods of statistics*, Princeton Univ. Press, Princeton N.J., 1946; MR 8, 39; pp. 57 and 89]. Conditions are given characterising kernels $K(x, t)$ such that if $F(t)$ belongs to R and $\psi(x) = \int_{-\infty}^\infty K(x, t)dF(t)$, then $\psi(x)$ belongs to C.

A. P. Guinand (Saskatoon, Sask.)

2854:

Mikusiński, J. Une simple démonstration du théorème de Titchmarsh sur la convolution. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 715–717. (Russian summary, unbound insert)

If f and g are continuous on $[0, T]$ and $\int_0^t f(u)g(t-u)du = 0$ for $0 \le t \le T$, then $f(t) = 0$ for $0 \le t \le t_1$ and $g(t) = 0$ for $0 \le t \le t_2$ where $t_1 + t_2 \ge T$. The author proves this theorem by first referring to a simple proof [see for instance Mikusiński, *Operational calculus*, Pergamon, New York, 1959; MR 21 #4333; pp. 20–22] for the special case $f = g$, $t_1 = t_2 \ge \frac{1}{2}T$, deducing from this special case that $f(t)$ and $t^n g(t)$ $(n = 0, 1, 2, \cdots)$ satisfy the same conditions as $f(t)$ and $g(t)$, and then using Lerch's theorem. In order to prove that $f(t)$ and $tg(t)$ satisfy the same conditions as $f(t)$ and $g(t)$, he defines α to be the sup of all numbers in $[0, 1]$ for which $\int_0^t f(t-u)g(u)du = 0$ in $[0, T]$ implies $\int_0^t f(t-u)ug(u)du = 0$ in $[0, \alpha T]$, shows by properties of the convolution that the last equation must hold in $[0, (1+\alpha^2)T/2]$ and concludes that $\alpha = 1$.

A. Erdélyi (Pasadena, Calif.)

2855:

Mehra, A. N. On Laplace transform. Bull. Calcutta Math. Soc. **51** (1959), 8–20.

The author gives some half dozen identities connecting chains of Laplace transforms. Let us write $f(t) \to \phi(p)$ if $p\int_0^\infty e^{-pt} f(t)dt = \phi(p)$. A typical result is the following. If $h \to f$, $g \to h$, $F \to \theta$, then

$$\int_0^\infty h(v)\theta(v)v^{-1}dv = \int_0^\infty \psi(s)g(s)ds,$$

where $\psi(s) = \int_0^\infty F(v)(v+s)^{-1}dv$. A variety of curious formulas result on specialization.

I. I. Hirschman, Jr. (Huntington, W. Va.)

2856:

Arya, Suresh Chandra. Some theorems connected with a generalized Stieltjes transform. Bull. Calcutta Math. Soc. **51** (1959), 39–47.

The author studies the result of iterating generalized Laplace and Stieltjes transforms. A large number of formulas are obtained which are recondite generalizations of the familiar result that if $f(s) = \int_0^\infty e^{-st}\phi(t)dt$ and if $\phi(s) = \int_0^\infty e^{-st}\psi(t)dt$ then $f(s) = \int_0^\infty (s+t)^{-1}\psi(t)dt$. These formulas are too complicated to reproduce here.

I. I. Hirschman, Jr. (Huntington, W. Va.)

2857:

Arya, Suresh Chandra. On two generalized Laplace transforms. Boll. Un. Mat. Ital. (3) **14** (1959), 307–317.

The author considers generalizations of the Laplace transform, such as

$$f(s) = \int_0^\infty e^{-st/2}(st)^{-k} W_{k,m}(st)da(t)$$

(which reduces to the Laplace transform when $k \pm m = \frac{1}{2}$). He gives two theorems of the form: If this integral converges for s_0, it does so for all s with $\operatorname{Re}(s) > \operatorname{Re}(s_0)$ provided k, m satisfy certain simple conditions. He also finds corresponding necessary conditions regarding the behavior of $a(t)$ for small t, and for large t.

T. E. Hull (Vancouver, B.C.)

2858:

Saxena, R. K. A study of the generalized Stieltjes transform. Proc. Nat. Inst. Sci. India. Part A **25** (1959), 340–355.

The generalized Stieltjes transform of $f(t)$ is

$$\phi(p) = p\int_0^\infty (p+t)^{-\lambda} f(t)dt.$$

The author evaluates some examples (none of which is new), and formulates some integral identities showing the connection of this transform with other integral transforms. *A. Erdélyi* (Pasadena, Calif.)

2859:

Barrucand, Pierre. Fonctions thêta et transformations de Fourier et de Mellin. C.R. Acad. Sci. Paris **250** (1960). 1783–1784.

It is shown here that the Mellin transform of $\vartheta_0(k' \log x/\pi k)/(1+x)$ is $\vartheta_1'(0)/\vartheta_1(s)$. Some other Fourier and Mellin transforms connected with this are also deduced.
A. P. Guinand (Saskatoon, Sask.)

2860:
Stanković, Bogoljub. Invariants d'une classe générale de transformations intégrales singulières. Univ. Beograd. Godišnjak Filozof. Fak. Novi Sad **2** (1957), 357–372. (Serbo-Croatian. French summary)

Let $A(D)=\sum a_\nu(D-a)^\nu$ be a function of the complex variable D. Now, taking abstraction of the nature of D, the author calls $A(D)$ an analytic operator and defines $A(D)f(z)=\sum a_\nu e^{az}[e^{-az}f(z)]^{(\nu)}$ for every $f(z)$ for which the series on the right does converge in some region of the z-plane.

He formally applies such analytic operators to find invariants of $\int K(t-x)f(t)dt$ and $\int K(tx)f(t)dt$, under conditions which are general enough to include the known ones for the Laplace-transformation of $t^{\alpha-1}+\lambda t^{-\alpha}\cdot\Gamma(\alpha)$.
V. Vučković (Belgrade)

2861:
Wunsch, G. Die Heavisidesche Operatorenrechnung in neuer Begründung. Beitrag zur Lösungstheorie linearer Differentialgleichungen. Hochfrequenztech. Elektroak. **69** (1960), 133–139.

2862:
Mikusiński, J. Sur la convolution par $\exp(t^2)$. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 669–671. (Russian summary, unbound insert)

The author proves the following theorem: If $f(t)$ and $g(t)=\int_0^t f(t-u)\exp u^2du$ possess Laplace transforms, then $f(t)=0$ a.e. on $(0, \infty)$. As a consequence, it is impossible to represent $\exp t^2$ as a convolution quotient [Mikusiński, *Operational calculus*, Pergamon, New York, 1959; MR **21** #4333] of functions possessing Laplace transforms.
A. Erdélyi (Pasadena, Calif.)

2863:
Sahnovič, L. A. Limiting values of a multiplicative integral. Ukrain. Mat. Ž. **11** (1959), 275–286. (Russian. English summary)

Let J be a diagonal matrix with elements $+1$ or -1. The author considers limiting values of the multiplicative integral

$$W(\lambda) = \int_a^b \exp\left(i\frac{dE(t)J}{t-\lambda}\right),$$

where $E(t)=\int_a^t \beta^2(s)ds$; $\beta^2(s)$ being a Hermitian positive matrix function. The result is that the limiting values

$$W^+(\sigma) = \lim_{\tau\to 0+} W(\sigma+i\tau), \quad W^-(\sigma) = \lim_{\tau\to 0-} W(\sigma+i\tau)$$

exist a.e. and are given in terms of certain multiplicative integrals. These results are closely connected with previous results of the author [Uspehi Mat. Nauk **12** (1957), no. 3 (75), 205–210; MR **19**, 937] where he assumed $J=\pm I$.
A. Devinatz (Princeton, N.J.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 2689, 2884.

2864:
Rakovščik, L. S. Integral equations with almost difference kernels. Dokl. Akad. Nauk SSSR **133** (1960), 752–755 (Russian); translated as Soviet Math. Dokl. **1**, 906–909.

By the integral equations mentioned in the title the author means equations of the type

$$a(t)\varphi(t)+\int_{-\infty}^{\infty} k(t, t-\tau)\varphi(\tau)d(\tau) = f(t).$$

With the aid of the Fourier transformation he states a number of theorems in regard to this equation and its Fourier transform. No proofs or derivations are given.
H. P. Thielman (Oxnard, Calif.)

2865:
Ringrose, J. R. On the Neumann series of an integral operator. Proc. London Math. Soc. (3) **10** (1960), 31–52.

The main theorem is as follows. Let $1\leq p<\infty$, $p^{-1}+q^{-1}=1$. Let $k(x, y)$ be a Lebesgue measurable function defined on $a\leq x, y\leq b$ and such that the double-norm $(\int_a^b dx\{\int_a^b |k(x, y)|^q dy\}^{p/q})^{1/p}$ is finite. Consider the conditions: (i) the corresponding integral operator K in $L_p(a, b)$ is quasi-nilpotent; (ii) there exists a bounded non-negative measurable function $\varphi(x)$ on (a, b) such that $k(x, y)=0$ at almost all points (x, y) where $\varphi(x)\leq\varphi(y)$. Then (ii) implies (i), and whenever $k(x, y)\geq 0$ on $a\leq x, y\leq b$, then (i) implies (ii). The case that $\varphi(x)=x$ gives rise to a Volterra integral operator.

From the other theorems proved we mention: If p and $k(x, y)$ are as above and if, in addition, $k(x, y)\geq 0$ on $a\leq x, y\leq b$ and $\min\{k(x, y), k(y, x)\}$ is strictly positive on a set of positive measure, then $k(x, y)$ has a positive eigenvalue ρ such that any other eigenvalue λ satisfies $|\lambda|\leq\rho$, and such that corresponding to ρ there exists a non-negative eigenfunction (not almost everywhere zero). If $k(x, y)$ is almost everywhere strictly positive, then ρ has multiplicity unity, the corresponding eigenfunction is almost everywhere strictly positive, and $|\lambda|<\rho$ for any other eigenvalue. This extends Jentzsch's theorem.
A. C. Zaanen (Pasadena, Calif.)

2866:
Gol'dengeršel', È. I. Growth of the solutions of systems of Volterra integral equations. Dokl. Akad. Nauk SSSR **125** (1959), 19–22. (Russian)

Author continues his investigation of the dependence on λ and f of the growth of solutions of the equation

$$(V-\lambda I)\phi = \int_0^x K(x, y)\phi(y)dy-\lambda\phi(x) = f(x)$$

as $x\to\infty$. See same Dokl. **124** (1959), 1195–1198 [MR **21** #279] for earlier results and definitions. As before, the fundamental idea is to study the action of V on the function spaces C_α consisting of those functions f on $[0, \infty)$ for which $\|f\|_\alpha=\sup_{0\leq x}\|f(x)\|\exp(-\alpha x)$ is finite. Exponents α considered are all $\alpha\geq\alpha_0$, where α_0 is arbitrary but fixed throughout; the kernel K is assumed to satisfy $\|K(x, y)\|\leq N\exp\nu(x-y)$ for $0\leq y\leq x<\infty$.

Discussion centers on two auxiliary functions: (1) $\gamma(V, \lambda)$, defined as the infimum of those $\alpha \geqq \alpha_0$ for which

$$\int_0^x \|\Gamma_{X'}(x, y)\| \exp(-\alpha(x-y))dy$$

is bounded ($\Gamma_{X'}$ denotes the kernel of $-\lambda I - \lambda^2 R_\lambda(V)$); and (2) $\tau_\lambda(V, \alpha)$, defined as the infimum of those ω for which $R_\lambda(V)C_\alpha \subset C_\omega$. Partial list of results: For λ fixed, τ_λ is a continuous monotone increasing function of α. Also $\tau_\lambda(\alpha) = \alpha$ for $\alpha > \gamma(\lambda)$. The function $\gamma(V, \lambda)$ satisfies $\gamma(V, \lambda) \leqq \nu + N/|\lambda|$, and is upper semi-continuous in λ for fixed V and (in a suitable sense) upper semi-continuous in V for fixed λ. If $\|K(x, y)\| \exp(-\alpha_0(x-y))$ is bounded and if $\tilde{K}$ is another kernel (defining the Volterra operator $\tilde{V}$) such that $\tilde{K}(x, y) \exp(-\alpha_0(x-y)) \to 0$ as $y \leqq x \to \infty$, then $\gamma(V, \lambda) = \gamma(V + \tilde{V}, \lambda)$ holds identically. Other conditions insuring invariance of γ under perturbation are given. The computation of $\gamma(g(V), \lambda)$ in terms of $\gamma(V, \lambda)$ is undertaken (g denotes a regular function on the spectrum of V with $g(0) = 0$). Several cases, in particular the important case $K(x, y) = k(x-y)$, are considered in which γ can be effectively computed. *A. Brown* (Houston, Tex.)

2867:

Picone, Mauro. Sull'equazione integrale non lineare di Volterra. Ann. Mat. Pura Appl. (4) **49** (1960), 1–10. (English summary)

A new theorem is given extending the class of non-linear Volterra integral equations whose solutions exist and are unique. The main result is incorporated in theorem B (p. 4). However, the notations involved are too detailed to be included in this abstract. The integral equation considered is of the form $u(x) = f(x) + \int_{x_0}^x K[x, y, u(y)]dy$, where $K(x, y, u) \equiv H(x, y, u)u$, H being a square matrix satisfying a type of Lipschitz condition. The approximating equations used in the existence proof are of the form

$$u^{(i)}(x) = f(x) + \int_{x_0}^x H[x, y, u^{(i-1)}(y)]u^{(i)}(y)dy \qquad (i = 1, 2, \cdots).$$

The author shows by means of an example that the classical result on non-linear Volterra equations (theorem A, on p. 3) does not suffice to insure the existence and uniqueness. *I. A. Barnett* (Cincinnati, Ohio)

2868:

Przeworska-Rolewicz, D. Sur les systèmes d'équations intégrales singulières pour des lignes fermées. Studia Math. **18** (1959), 247–268.

The author considers systems of non-linear integral equations of the form

$$(1) \qquad u(t) = \lambda \int_L \frac{K[t, \tau, u(\tau)]}{\tau - t} d\tau,$$

where L is a set of arcs. Equation (1) is a special case of the more general functional relation

$$(2) \qquad u(t) = \lambda M[t, u(t), \rho \int_L \frac{K[t, \tau, u(\tau)]}{\tau - t} d\tau],$$

where M is some non-linear functional. After some preliminary remarks on Cauchy integrals, the existence of solutions of (2) and hence also (1) is established by means of the Schauder fixed-point theorem for suitable K and M. It is also shown that for λ sufficiently small the solution can be obtained by a method of successive approximations. *R. C. MacCamy* (Pittsburgh, Pa.)

2869:

Vasilenko, O. Yu. Concerning a certain integral equation. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 942–944. (Ukrainian. Russian and English summaries)

Author's summary: "In this paper the author proves the existence and uniqueness of the positive solution of a non-linear integral equation which is dealt with in the theory of the non-steady flow of ground waters."

2870:

Vasilenko, O. Yu. Solution of an integral equation. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 1184–1188. (Ukrainian. Russian and English summaries)

Author's summary: "In this paper the author offers two iteration methods for the solution of a non-linear integral equation which is encountered in the theory of the non-steady flow of ground waters. The convergence of successive approximations, as obtained by the iteration methods, towards the solution is also proved."

2871:

Karlin, S.; Szegö, G. On certain differential-integral equations. Math. Z. **72** (1959/60), 205–228.

The main differential-integral equation considered is $\prod_{k=1}^n (D + \lambda_k) f(x) = \lambda_1 \cdots \lambda_n \int_0^\infty f(x+t)dH(t)$, subject to the condition that $\lim_{x\to\infty} f(x)$ exist finitely, D being the derivative operator d/dx, $\lambda_1, \cdots, \lambda_n$ positive numbers, $x \geqq 0$, $H(t)$ a distribution function on $[0, \infty)$ with finite positive second moment. Such a differential-integral equation arises in models of stochastic processes [see K. J. Arrow, S. Karlin, and H. Scarf, *Studies in the mathematical theory of inventory and production*, Stanford Univ. Press, Stanford, Calif., 1958; MR **20** #757; Chap. 14]. It is shown that the only solutions are linear combinations of exponential solutions $e^{-\alpha x}$, with $\Re(\alpha) > 0$, forming an n-dimensional manifold if $\sum_1^n \lambda_k^{-1} < \int_0^\infty t dH(t)$, and an $(n-1)$-dimensional manifold if $\sum_1^n \lambda_k^{-1} \geqq \int_0^\infty t dH(t)$. Slight changes in this result occur when the λ_k are any nonzero numbers. If x ranges over $-\infty < x < \infty$ and both limits $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are required to exist finitely, then $f(x) \equiv$ constant is the only solution of the equation. An extension to semi-groups leads to the consideration of the equation

$$\prod_{k=1}^n (A + \lambda_k) f(x) = \lambda_1 \cdots \lambda_n \int_0^\infty (T_t f)(x) dH(t),$$

with x on I, Γ a linear space of bounded functions on I, and $T_t f$ a semi-group of bounded linear transformations on Γ to Γ, with A the infinitesimal generator of T_t—formally $T_t \sim e^{tA}$. Continuity of T_t in t, pointwise in x on I, and continuity at $t = 0$, with T_0 the identity, is assumed. In addition to the case already considered, mention is made of the transformations

$$T_t f(x) = (2\pi t)^{-1/2} \int_{-\infty}^\infty \exp(-(x-y)^2/2t) f(y) dy$$

with $I=(-\infty, \infty)$ and $Af=\frac{1}{2}f''$, which is related to the heat equation $u_t=\frac{1}{2}u_{xx}$, with $u(0, x)=f(x)$ on I. Also n-dimensional extensions of the heat equation, as well as a discrete case where $I=1, 2, \cdots, n, \cdots$ are considered. By setting $w(t, x)=(T_tf)(x)$ for $t>0$, x on I, the solution of the functional equation can be formally reduced to that of

$$\prod_{k=1}^{n}(D_x+\lambda_k)w(s, x) = \lambda_1\cdots\lambda_n\int_0^\infty w(s+t, x)dH(t).$$

The condition that $\lim_{s\to\infty} w(s, x)$ exist pointwise in x needs to be interpreted in each special case, for the heat equation it is equivalent to the existence of $\lim_{t\to\infty}(2t)^{-1}\times\int_{-t}^{t}f(x)dx$. If $w(s, x)=\sum\gamma_l(x)\exp(-\alpha_l s)$, with $\Re(\alpha_l)>0$, then solutions of the original functional equation are linear combinations of $\nu_l(x)$, and these latter satisfy the equations $A\gamma_l(x)=-\alpha_l\gamma_l(x)$. *T. H. Hildebrandt* (Providence, R.I.)

FUNCTIONAL ANALYSIS

See also 2747, 2817, 2849, 2871, B3477.

2872:

Monna, A. F. Espaces localement convexes sur un corps valué. Nederl. Akad. Wetensch. Proc. Ser. A **62**= Indag. Math. **21** (1959), 391–405.

Extrait de la préface: "Dans l'article présent il s'agit d'appliquer les résultats d'un article antérieur [Monna, mêmes Proc. **61** (1958), 528–539; MR **21** #5128] dans l'étude des espaces vectoriels topologiques sur un corps K muni d'une valuation non-archimédienne propre. On donne des propriétés des semi-normes non-archimédiennes continues, et au moyen des ensembles K-convexes on peut définir les espaces localement K-convexes en analogie avec les espaces localement convexes réels. On montre que tout espace localement K-convexe séparé est isomorphe à un sous-espace d'un produit d'espaces de Banach non-archimédiens. Enfin on étudie la possibilité d'un prolongement des transformations linéaires et des fonctionnelles linéaires (théorème de Hahn-Banach)."

G. K. Kalisch (Minneapolis, Minn.)

2873:

Iséki, Kiyoshi. A class of quasi-normed spaces. Proc. Japan Acad. **36** (1960), 22–23.

$\|\lambda x\|=|\lambda|^r\|x\|$, where the valuation $\lambda\to|\lambda|$ is non-archimedean. *L. Gillman* (Rochester, N.Y.)

2874:

Campos Ferreira, Jaime. Les espaces de Schwartz et les espaces d'applications linéaires continues. Portugal. Math. **18** (1959), 1–32.

The author considers, after Sebastião e Silva [Rend. Mat. e Appl. (5) **14** (1955), 388–410; MR **16**, 1122], direct and inverse limits of normed spaces, where the mappings $f_{\alpha\beta}$ which define the limits are compact (completely continuous): he calls such spaces (S_2) and (S_1) respectively, and studies the spaces $L_b(E, F)$ of continuous mappings of E into F, when E and F belong to one of these classes of spaces. When E is (S_2) and F is (S_1), $L_b(E, F)$ appears naturally as an inverse limit of spaces $L_b(E_n, F_n)$, and the author easily proves that its topology coincides with the inverse limit topology. When E is (S_1) and F is (S_2), $L_b(E, F)$ becomes a direct limit, but here there is no obvious identification of topologies. Finally, when both E and F are (S_2), or both (S_1), $L_b(E, F)$ is, at least algebraically, an inverse limit of spaces which are direct limits of normed spaces; such spaces the author calls (S_{12}). All the results are easy consequences of known theorems and lie close to the surface.

J. Dieudonné (Paris)

2875:

Musielak, J.; Orlicz, W. Some remarks on modular spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 661–668. (Russian summary, unbound insert)

As the title shows, this paper, consisting of six parts, contains literally only fragmental remarks to the previous paper by the authors in Studia Math. **18** (1959), 49–65 [MR **21** #298]. Here, admitting the case $\rho(x)=0$ for $x\neq 0$, the modular is generalized, calling it pseudomodular. In parts 1 and 2 a topology is induced by ρ in the same way as in the previous paper. In part 3, the quotient of the space by $\{x: \rho(kx)=0 \text{ for all } k\}$ is considered. The results mentioned in parts 4 and 5 are already available in the reviewer's book, *Modulared semi-ordered linear spaces* [Maruzen, Tokyo, 1950; MR **12**, 420]. The last part 6 can be considered as a remark to the reviewer's paper in J. Math. Soc. Japan **5** (1953), 29–49 [MR **15**, 442].

H. Nakano (Kingston, Ont.)

2876:

Ishii, Jyun. On equivalence of modular function spaces. Proc. Japan Acad. **35** (1959), 551–556.

Modulared function space L_Φ defined by H. Nakano on a measure space (Ω, μ) is the space of measurable functions $x(\omega)$ such that

$$m_\Phi(\alpha x) = \int_\Omega \Phi(\alpha|x(\omega)|, \omega)d\mu(\omega) < \infty$$

for some $\alpha>0$, where $\Phi(\xi, \omega)$ not only is a convex function of $\xi\geq 0$ as in the case of Orlicz spaces but also depends measurably on $\omega\in\Omega$.

The author proves that L_Φ is included in another L_Ψ if and only if there exist $k, K>0$ and an integrable function $c(\omega)$ such that $\Psi(k\xi, \omega)\leq K\Phi(\xi, \omega)+c(\omega)$ a.e. in Ω, for all $\xi\geq 0$ in the case of non-atomic μ, or for all $0\leq\xi\leq d(\omega)$ in the atomic case, where $d(\omega)$ is a function which satisfies $d(\omega)\geq|x(\omega)|$ at all except finite number of ω whenever $m_\Phi(x)<\infty$. *I. G. Amemiya* (Tokyo)

2877:

Sasaki, Masahumi. On semi-lower bounded modulars. J. Fac. Sci. Hokkaido Univ. Ser. I **14**, 114–124 (1959).

Let R be a universally continuous semi-ordered linear space, i.e., a complete vector lattice, and let m be a modular on R, i.e., a mapping of R into the extended real number system such that: (i) $0\leq m(x)\leq\infty$ for all $x\in R$; (ii) $m(\xi x)=0$ for all $\xi\geq 0$ implies $x=0$; (iii) for any $x\in R$, there exists a number $\alpha>0$ such that $0\leq m(\alpha x)<\infty$; (iv) for all $x\in R$, $m(\xi x)$ is a convex function of ξ; (v) $|x|\leq|y|$ implies $m(x)\leq m(y)$; (vi) $x\wedge y=0$ implies $m(x+y)=m(x)+m(y)$; $0\leq x_\lambda\uparrow x$ $(\lambda\in\Lambda)$ implies $m(x)=\sup(m(x_\lambda), \lambda\in\Lambda)$.

A modular is called finite if $m(x)<\infty$ for all $x\in R$. A modular is said to be monotone complete if for any sub

family $(x_\lambda, \lambda \in \Lambda)$ of R which is directed upwards such that $\sup(m(x_\lambda), \lambda \in \Lambda)$ is finite, then for some $x \in R$, $x_\lambda \uparrow x$ $(\lambda \in \Lambda)$. A modular m is said to be semi-upper bounded if for every $\varepsilon > 0$ there exists a number $\gamma = \gamma(\varepsilon) > 0$ such that $m(x) \geq \varepsilon$ implies $m(2x) \leq \gamma m(x)$. A modular m is said to be semi-lower bounded if for every $\varepsilon > 0$, there exist numbers $\alpha = \alpha(\varepsilon)$ and $\gamma = \gamma(\varepsilon)$ such that $1 < \alpha < \gamma$ and $m(x) \geq \varepsilon$ implies $m(\alpha x) \geq \gamma m(x)$.

Let $\bar{R}^m$ be the vector space of all universally continuous linear functionals. (A linear functional L on R is called universally continuous if $x_\lambda \downarrow 0$ $(\lambda \in \Lambda)$ implies $\inf(|L(x_\lambda)|, \lambda \in \Lambda) = 0$.) On $\bar{R}^m$ the conjugate modular $\overline{m}$ is defined as follows: $\overline{m}(x) = \sup(\overline{x}(x) - m(x),\ x \in R)$ for all $\overline{x} \in \bar{R}^m$. Then $\overline{m}$ is a modular on $\bar{R}^m$ and is monotone complete.

The author shows that the notions semi-upper bounded and semi-lower bounded are dual notions, i.e., if m is semi-lower bounded, then $\overline{m}$ is semi-upper bounded and if m is semi-simple and semi-upper bounded, then $\overline{m}$ is semi-lower bounded. This result in combination with the result, recently obtained by I. Amemiya [same J. **13** (1956), 60–64; MR **18**, 491], that every monotone complete and finite modular on a universally continuous semi-ordered linear space R is semi-upper bounded if R has no atoms, give rise to a variety of interesting results of which we only quote the following: If R has no atoms and m is uniformly convex, then m is semi-upper bounded as well as semi-lower bounded.

W. A. J. Luxemburg (Pasadena, Calif.)

2878:

Hunter, Larry C. On induced topologies in quasi-reflexive Banach spaces. Proc. Amer. Math. Soc. **11** (1960), 161–163.

Let π denote the natural embedding of a Banach space X into its second conjugate space X^{**}. The space X is called quasi-regular of order n [Civin and Yood, Proc. Amer. Math. Soc. **8** (1957), 906–911; MR **19**, 756] if $X^{**}/\pi X$ is n-dimensional. Let X be such a space. The author considers decompositions of X^* of the form $X^* = Q \oplus R$ with Q total and R n-dimensional. Each such decomposition yields a topology in which a suitably renormed X has a compact unit ball; in fact, X is equivalent to Q^*.

P. Civin (Gainesville, Fla.)

2879:

Kadec', M. Ĭ. On the connection between weak and strong convergence. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 949–952. (Ukrainian. Russian and English summaries)

Author's summary: "Let Γ be a linear total set of linear functionals defined in a separable Banach space E and satisfying the following condition: there exists such a number $\eta > 0$ that for every $x \in E$ an $f \in \Gamma$ can be found satisfying the inequality

$$|f(x)| \geq \eta \cdot \|f\| \cdot \|x\|.$$

If E is a space which has a basis, Γ may be a linear hull of functionals, which are biorthogonal to the elements of the basis. If $E = E_1^*$ is a separable conjugate Banach space, Γ may be a set of weak continuous linear functionals, i.e., functionals generated by the elements of the space E_1.

"Theorem 1: A new norm $\| \ \|^*$ may be defined in E, equivalent to the usual one and causing the conditions (a) $\lim_{n\to\infty} f(x_n - x) = 0$ for all $f \in \Gamma$ and (b) $\lim_{n\to\infty} \|x_n\|^* = \|x\|^*$ to result in strong convergence: $\lim_{n\to\infty} \|x_n - x\|^* = 0$. With the aid of the theorem 1 we may prove theorem 2: All separable conjugate Banach spaces are homeomorphic."

2880:

Cagliardo, Emilio [Gagliardo, Emilio]. Interpolation d'espaces de Banach et applications. II. C. R. Acad. Sci. Paris **248** (1959), 3388–3390.

[For Part I see same C. R. **248** (1959), 1912–1914; MR **21** #1515].

In this note is given a generalization of the construction given in the prior note. For convenience of exposition the reviewer has introduced a few notations not in the paper. Let A and B be Banach spaces, and let U be a closed linear one-to-one mapping with domain in A and range in B. Let Z be the graph of $-U$ in $A \times B$. There is a natural algebraic isomorphism $a \to \bar{a}$ between A and a subspace of $E = (A \times B)/Z$. Likewise for B. Every element u of E can be expressed in the form $u = \bar{a} + \bar{b}$. E is a Banach space with $\|u\|$ defined as $\inf(\|a\| + \|b\|)$ for $\bar{a} + \bar{b} = u$.

To construct intermediary spaces, consider sets M of points (x, y) in the plane, with $x \geq 0$, $y \geq 0$, and having the properties: M is convex and with (x, y) also contains $(x+h, y+k)$ for all nonnegative h, k. Multiplication of M by a positive constant, and addition of several M's, are defined in obvious ways. There is then introduced a class of functions F with nonnegative values, defined on the class of M's, and having certain properties. For certain constants c_0, c_1, c_2 it is required that $F[\lambda M] \leq c_0 \lambda F[M]$ $(\lambda > 0)$, $F[\sum_i M_i] \leq c_1 \sum_i [M_i]$, $F[M] \geq c_2 \inf(x+y)$ for $x, y \in M$, and a certain quasi-continuity property.

If $u \in E$, let $M(u) = \{(x, y)$: there exist $a \in A$, $b \in B$ with $u = \bar{a} + \bar{b}$, $\|a\| \leq x$, $\|b\| \leq y\}$. Then, for given F, let $C = \{u \in E : F[M(u)] < +\infty\}$, $\|u\|_C = \inf \sum_i F[M(u_i)]$, $\sum u_i = u$. Then C is an intermediary space. The family of C's includes A, B and E itself.

A. E. Taylor (Los Angeles, Calif.)

2881:

Gagliardo, Emilio. Interpolation d'espaces de Banach et applications. III. C. R. Acad. Sci. Paris **248** (1959), 3517–3518.

In this note the author uses the framework of note II to deal with questions of extension of linear and quasilinear mappings. Suppose that A, B, U, E are as in the preceding review, and let A', B' be another pair of Banach spaces, with a corresponding U', E' as in the first case. For convenience we henceforth regard A and B as embedded in E by the natural isomorphisms. Likewise for A', B' in E'. If T_1 and T_2 are continuous linear mappings of A into A' and B into B', respectively, which coincide in the intersection of A and B, then there is a simultaneous extension of T_1 and T_2 as a continuous linear mapping of E into E'. This situation is generalized to cover a kind of quasilinearity of mappings. If C is an intermediary space between A and B, defined by a functional F as in note II, and if C' is the corresponding intermediary space between A' and B', then from certain specified hypotheses on a quasilinear mapping of E into E', with $T(A) \subset A'$, $T(B) \subset B'$, one can infer that $T(C) \subset C'$ and obtain an inequality $\|T(u)\|_{C'} \leq J\|u\|_C$ (J a positive constant).

The author indicates that this theory of interpolating

spaces has many applications to spaces of functions. The S. L. Sobolev spaces can be constructed by this theory. Relations to theorems of M. Riesz, J. Marcinkiewicz, E. M. Stein and G. Weiss are mentioned.

A. E. Taylor (Los Angeles, Calif.)

2882:

Il'in, V. P. Some integral inequalities for differentiable functions of several variables. Dokl. Akad. Nauk SSSR **129** (1959), 1214–1217. (Russian)

The author considers a function $f(x)=f(x_1, \cdots, x_n)$ defined, with continuous partial derivatives of order l, on a region D in Euclidean n-dimensional space E_n, and writes

$$D^j f = (\sum |\partial^j f/\partial x_{i_1} \cdots \partial x_{i_j}|^2)^{1/2} \quad (j = 0, 1, \cdots, l),$$

where the summation extends over all sets of positive integers $i_1, \cdots, i_j$ with sum j. He supposes that every point Y of D is the vertex of some spherical sector of fixed radius and form that lies wholly in D, and denotes by H the greatest admissible radius. He gives, without proofs, several inequalities involving the norm $N_p(l, \alpha; n; D)$ of $D^l f(X)|X-Z|^\alpha \chi(D)$ on the L_p-space of functions on E_n, where $\chi(D)$ is the characteristic function of the set D and Z is a fixed point in D. For instance, his theorem 2 states sufficient conditions (too elaborate for reproduction here) upon k, m, p, q, α, β, ν, and Z, under which

$$h^{k-(m/q)} N_q(k, -\beta; m; S_m) \leqq K_1 N_p(l, \alpha; n; D) h^{l-(n/p)-\alpha-\beta} + K_2 N_\nu(0, 0; n; D) h^{-n/\nu} b^{-\beta}$$

for every positive $h \leqq H$; where $1 \leqq m \leqq n$. S_m is a bounded region in the section of D by an m-dimensional hyperplane E_m, $b=h$ if $\beta \geqq 0$, but b is equal to $d=\sup|Y-Z|$ for points Y in S_m if $\beta<0$, and K_1, K_2 are constants independent of f, h; and d. Theorem 3 is similar, but with $|X-Z|$ replaced by the distance ρ of X from E_m, and S_m by the subset of D on which $\rho \leqq d$. The author states that all his theorems are based on inequalities for integrals of the type of Hilbert's inequality and especially on analogues and generalizations of a two-parameter inequality due to G. H. Hardy and J. E. Littlewood [Math. Z. **27** (1928), 565–606].

H. P. Mulholland (Exeter)

2883:

Gross, Leonard. Integration and nonlinear transformations in Hilbert space. Trans. Amer. Math. Soc. **94** (1960), 404–440.

The author proves, by a method based on finite-dimensional approximations, a theorem on change of variable in a Wiener-Gauss integral in Hilbert space. The (non-linear) change of variable is required to be of the form $1+\varphi$, where φ has a Fréchet derivative of trace-class, and satisfies various other appropriate continuity assumptions.

J. T. Schwartz (New York)

2884:

Guy, Roland. Équations intégrales du type de Volterra dans un espace de Hilbert. C. R. Acad. Sci. Paris **249** (1959), 2710–2712.

Pour l'équation "intégrale" $x(t)=x_0(t)+\int_{t_0}^t \mathscr{F}(\tau)x(\tau)\,d\tau$ avec $x(t)$ à valeurs dans un espace de Banach, $\mathscr{F}(\tau)$ étant une famille d'opérateurs non-bornés mais vérifiant certaines conditions très restrictives et difficiles à être appliquées effectivement, on démontre l'existence des solutions par des approximations successives.

S. Zaidman (Bucharest)

2885:

Zaidman, Samuel. Sur un théorème de I. Miyadera concernant la représentation des fonctions vectorielles par des intégrales de Laplace. Tôhoku Math. J. (2) **12** (1960), 47–51.

Miyadera [same J. (2) **8** (1956), 170–180; MR **18**, 748] has given necessary and sufficient conditions that a function $f(s)$, defined for $s>0$ and with values in a reflexive Banach space X, be represented as the Laplace transform of a function in the Bochner class $B_\infty([0, \infty], X)$. The author shows by a counter-example that Miyadera's conditions are no longer sufficient if the condition on X of reflexivity is relaxed to one of weak sequential completeness.

P. G. Rooney (Toronto)

2886:

Goffman, Casper; Waterman, Daniel. Basic sequences in the space of measurable functions. Proc. Amer. Math. Soc. **11** (1960), 211–213.

A simple proof and applications of the following result. If $\varphi_1, \varphi_2, \cdots$ is a fundamental set of measurable functions on [0, 1] (i.e., if each measurable function is a limit of a sequence of linear combinations of the φ_n), then also $\varphi_p, \varphi_{p+1}, \cdots$ $(p \geqq 1)$ is a fundamental set.

G. G. Lorentz (Syracuse, N.Y.)

2887:

Fréchet, M. Supplément à l'article "Sur deux problèmes d'analyse non résolus". Colloq. Math. **7** (1959/60), 201–204.

A proof in the last section of the paper named in the title [same Colloq. **6** (1958), 33–40; MR **20** #7208] is corrected. In addition, the second problem proposed there is discussed further. For any system $Z(x, y; p, q)$, where x, y are in the metric space Γ of continuous curves in R_3 and p, q are non-negative real numbers with $p+q=1$, $\delta \in \Gamma$ is said to be a mean (T) if distances to every $a \in \Gamma$ satisfy the inequality: $(\delta, a) \leqq p(x, a)+q(y, a)$. It is shown that the problem could be settled by proving that either of the following questions has an affirmative answer: (1) Is there a system $Z(x, y; p, q)$ without a mean (T)? (2) Does every system $Z(x, y; p, q)$ have exactly one mean (T)?

C. W. Kohls (Rochester, N.Y.)

2888:

Kaplan, Samuel. The second dual of the space of continuous functions. II. Trans. Amer. Math. Soc. **93** (1959), 329–350.

The author's study [same Trans. **86** (1957), 70–90; MR **19**, 868] (denoted by I below) of the Banach lattice C of real continuous functions on a compact Hausdorff space and of its first and second duals L and M is here continued. The paper begins with a brief self-contained exposition of the necessary parts of the Nakano theory [H. Nakano, *Modulared semi-ordered linear spaces*, Maruzen, Tokyo, 1950; MR **12**, 420] of the bounded linear functionals on vector lattices which are continuous for order convergence. This theory is then applied to the above spaces to obtain the duality of L and M with respect to this continuity,

various properties of their order-closed sub-vector-lattices, and a concrete realization as a sub-vector-lattice in M of the dual of any order-closed sub-vector-lattice in L. Characterizations are given of (i) the subspaces in L of the form $\mathscr{L}^1(\mu)$, μ a Radon measure, and those in M of the form $\mathscr{L}^\infty(\mu)$, (ii) the relatively compact subsets of L under the weak topology $w(L, M)$, (iii) the Mackey topology $\tau(M, L)$ on M. Turning to the spaces of bounded linear functionals on specific subspaces of M (the Baire, Borel, and universally integrable subspaces defined in I), a definition of regularity is introduced and studied in such spaces of functionals. The set of regular functionals on these spaces is shown to be L. Finally the space of bounded linear functionals on M is considered briefly.

W. R. Transue (Gambier, Ohio)

2889:

Royden, H. L. On a paper of Rogosinski. J. London Math. Soc. **35** (1960), 225–228.

Let $C(X)$ denote the space of complex-valued continuous functions on a compact Hausdorff space X. The paper characterizes the bounded linear functionals F defined on a subspace S of $C(X)$ for which there is a maximal function, i.e., a function $f \in S$ such that $F(f) = \|F\| \, \|f\|$. For the case where X is the real interval [0, 1] this problem was considered by Rogosinski [Proc. London Math. Soc. (3) **6** (1956), 175–190; MR **17**, 987], and the present characterization in terms of Radon-Nikodym derivatives may be applied to obtain Rogosinski's result.

W. R. Transue (Gambier, Ohio)

2890:

Kolmogorov, A. N.; Tihomirov, V. M. ε-entropy and ε-capacity of sets in function spaces. Uspehi Mat. Nauk **14** (1959), no. 2 (86), 3–86. (Russian)

Let A be a compact non-empty subset of a metric space, and let $\varepsilon > 0$. If $N_\varepsilon(A)$ denotes the minimal number of points of an ε-net in A, and $M_\varepsilon(A)$ the maximal number of points in an ε-distinguishable subset B of A (B is ε-distinguishable if any two of its points are at a distance greater than ε from each other), then $H_\varepsilon(A) = \log N_\varepsilon(A)$ is the entropy of A, and $C_\varepsilon(A) = \log M_\varepsilon(A)$ is the capacity of A. These definitions are due to Kolmogorov [Dokl. Akad. Nauk SSSR **108** (1956), 385–388; MR **18**, 324], who proposed to measure the "massiveness" of A by the asymptotic behavior of $H_\varepsilon(A)$ for $\varepsilon \to 0$ and computed the entropies of several sets of functions. The present paper gives an exposition of this theory developed by the authors and by other Russian mathematicians. [Another such exposition with applications to the tabulation of functions has been given by A. G. Vituškin in the book, *Ocenka složnosti zadači tabulirovaniya*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959.] No applications are given except in Appendix I. The relation of the metric entropy discussed here to the probabilistic entropy and the theory of information remains that of an analogy.

The principal general fact about the functions $H_\varepsilon(A)$, $C_\varepsilon(A)$ is the inequality

$$(*) \qquad C_{2\varepsilon}(A) \leq H_\varepsilon(A) \leq C_\varepsilon(A).$$

This is repeatedly used for special sets A; to determine the asymptotic behavior of $H_\varepsilon(A)$, one has to find a lower bound of $H_\varepsilon(A)$ and an upper bound for $C_\varepsilon(A)$ which are sufficiently close to each other. If A is the set of real functions on $[a, b]$ which satisfy $|f(x)| \leq C$, $|f(x) - f(x')| \leq L|x - x'|$, then in the distance generated by the uniform norm for $[a, b]$, $H_\varepsilon(A) = (b-a)L\varepsilon^{-1} + \log(C\varepsilon^{-1}) + O(1)$. For a bounded subset A with interior points of an s-dimensional Banach space D_s, there exists a constant $\theta \geqq 1$ such that $N_\varepsilon(A) \sim \theta(\mu A/\mu S)\varepsilon^{-n}$, where μ is the Lebesgue measure and S is the unit ball in D_s. Most of the remaining estimates are less precise and give H_ε only up to a strong or a weak equivalence. (**) For the set A of real functions on a parallelepiped of dimension s which have continuous partial derivatives of order $j \leqq p$ bounded by fixed constants, the derivatives of order p satisfying a Hölder condition of order $0 < \alpha \leqq 1$ and again with a fixed constant, one has, in the uniform norm on the parallelepiped, $H_\varepsilon \asymp \varepsilon^{-(p+\alpha)/s}$. A similar but somewhat more precise result is given for the set of periodic functions with Weyl's derivative of order α, in the norm of the space L^2.

Further estimates concern analytic functions. If K is a simply-connected continuum of the complex plane and G an open set with $G \supset K$, if A consists of functions analytic in G and bounded by 1 in absolute value, then in the uniform norm on K, $H_\varepsilon(A) \sim \tau(K, G)(\log(1/\varepsilon))^2$ (Babenko and Erohin). The constant $\tau(K, G)$ can be determined by conformal mapping. A similar result holds for functions of several complex variables. For special K, G, one has better estimates. For example, for functions $f(z_1, \cdots, z_s)$, analytic and bounded by one for $|z_j| < r_j$, $r_j > 0$, $j = 1, \cdots, s$, in the uniform norm on the product of the circles $|z_j| \leqq 1$, $j = 1, \cdots, s$, one has

$$H_\varepsilon = 2\left(\log\frac{1}{\varepsilon}\right)^{s+1} [(s+1)! \prod_{j=1}^{s} \log r_j]^{-1} + O\left(\left(\log\frac{1}{\varepsilon}\right)^s \log\log\frac{1}{\varepsilon}\right).$$

For entire functions with $|f(z)| \leqq C \exp(\sigma|z|^p)$, in the uniform norm on $|z| \leqq 1$,

$$H_\varepsilon \sim p\left(\log\frac{1}{\varepsilon}\right)^2 \left[\log\log\frac{1}{\varepsilon}\right]^{-1},$$

and for 2π-periodic entire functions with $|f(z)| \leqq C \times \exp(\sigma|\mathrm{Im}\, z|^p)$, in the uniform norm on the real line, $H_\varepsilon \sim C(p, \sigma)(\log(1/\varepsilon))^{2-1/p}$ (actually, the last two results are given for functions of s complex variables). If the assumption of periodicity is dropped for the last class, and $H_\varepsilon^T(A)$ is the entropy of A in the uniform norm on the segment $[-T, T]$ of the real line, then

$$\liminf_{T\to\infty} \frac{1}{2T} H_\varepsilon^T(A) \sim \limsup_{T\to\infty} \frac{1}{2T} H_\varepsilon^T(A) \sim C_1(p, \sigma)\left(\log\frac{1}{\varepsilon}\right)^{2-1/p}.$$

The last section deals with entropies of sets of linear functionals which satisfy a Hölder condition.

Appendix I is a contribution to the 13th problem of Hilbert. In spite of its negative solution by Kolmogorov [see #2669 above], Hilbert's hypothesis is reasonable, if formulated in more general form: one cannot represent all functions of a large class by superpositions of functions of small classes. From this point of view, Kolmogorov's result may be interpreted to mean that classes of arbitrary continuous functions are so large that the number of variables is not an adequate measure for their largeness. For smooth functions the situation is different. With the set of functions of s variables described under (**) we associate "the degree of smoothness" $p = (p + \alpha)/s$; then the result (which generalizes a former theorem by

Vituškin [Dokl. Akad. Nauk SSSR **95** (1954), 701–704; MR **15**, 945]) is that not all functions of smoothness ρ_0 can be represented by finite superpositions of functions with $\rho > \rho_0$ and $p+\alpha \geqq 1$. The proof is by combination of a category and an entropy argument. Appendix II stands isolated in this paper; it deals with probabilistic entropy. An inequality of type (**) is obtained for signal transmission. {There are several misprints, especially in Appendix I.} *G. G. Lorentz* (Syracuse, N.Y.)

2891:

Brudnyĭ, Yu. A.; Timan, A. F. Constructive characteristics of compact sets in Banach spaces and ε-entropy. Dokl. Akad. Nauk SSSR **126** (1959), 927–930. (Russian)

The main content of this paper concerns connections between the ε-entropy $I_W(\varepsilon)$ of a set W [see Kolmogorov and Tihomirov, preceding review] and the approximation properties of W. Let W be contained in a separable Banach space F with a fundamental sequence $X=\{x_n\}$ of elements. If $E_n{}^X(x)$ is the measure of the best approximation of $x \in F$ by sums $\sum_{k=0}^n c_k x_k$, then the problem is to estimate $\mathscr{E}_n{}^X(W) = \sup_{x\in W} E_n{}^X(x)$ and $D_n(W) = \inf_X \mathscr{E}_n{}^X(W)$. The following results are formulated. Let W_X be the extension of $W \subset F$ consisting of all $x \in F$ with $\|x\| \leqq \mathscr{E}_0{}^X(W)$, $E_k{}^X(x) \leqq \mathscr{E}_k{}^X(W)$, $k=1, 2, \cdots$, and let $\varepsilon = \Phi_W(\eta)$ be the inverse function to $\eta = I_W(\varepsilon)$. Then $\mathscr{E}_{n-1}^X(W) \leqq 4\Phi_{W_X}(n+1)$. If F is a Hilbert space, and $\mathfrak{M}_k{}^\beta(F)$ consists of all $W \subset F$ for which the kth logarithm of the entropy satisfies $\log_k I_W(\varepsilon) \asymp (\log(1/\varepsilon))^\beta$, then $D_n(W)$ for all large n is greater than or equal to: (a) $\exp(-Cn^{1/(\beta-1)})$ for $k=0$, $\beta > 1$; (b) $\Phi_W(Cn(\log n)^{1/\beta})$ for $k=1$, $0<\beta\leqq 1$; (c) $\exp(-Cn^{1/\beta})$ for $k=1$, $\beta>1$; (d) $\Phi_W(n^2)$ for $k \geqq 2$. Here $C>0$ does not depend on n, and $W \in \mathfrak{M}_k{}^\beta(F)$. The proofs are said to depend on an inequality connecting $I_W(4\varepsilon)$, $n_W{}^X(\varepsilon)$ (the smallest integer n with $\mathscr{E}_n{}^X(W) \leqq \varepsilon$), and $d_n{}^X = \inf_{0\leqq\nu\leqq n}\{\|x_\nu\|^{-1} \inf_{c_k} \|x_\nu - \sum^* c_k x_k\|\}$ (with $k \neq \nu$, $k \leqq n$ in the sum $\sum^*$).

There are applications to the estimation of $D_n(W)$ and to the computation of entropies. For instance, if $1 \leqq p \leqq +\infty$, $0<\alpha\leqq 1$, and W is the set of all 2π-periodic functions f which satisfy in the L^p-norm $\|f(t)\| \leqq K$, $\|f^{(r)}(t+h) - 2f^{(r)}(t) + f^{(r)}(t-h)\| \leqq h^\alpha$, then $I_W(\varepsilon)$, taken in the L^p-norm, is between $C_1(1/\varepsilon)^{1/(r+\alpha)} \log(1/\varepsilon)$ and $C_2(1/\varepsilon)^{1/(r+\alpha)}$. For the s-dimensional case, the exponent is to be replaced by $s/(r+\alpha)$. *G. G. Lorentz* (Syracuse, N.Y.)

2892:

Bonsall, F. F. Semi-algebras of continuous functions. Proc. London Math. Soc. (3) **10** (1960), 122–140.

Let $C(E)$ denote the class of all continuous real-valued functions on a compact Hausdorff space E. A subset F of $C(E)$ is called a semi-algebra if $f+g$, fg, αf belong to F whenever f, g belong to F and α is a non-negative real number. A semi-algebra F is said to be of type n ($\geqq 0$) if $f^n/(1+f)$ belongs to F whenever f is in F. The author proves that a uniformly closed semi-algebra of type 0 is the class of all non-negative functions f in $C(E)$ that satisfy a family of equations of the form $f(s) = f(s')$ for some s, s' in E. He also studies uniformly closed semi-algebras of type 1 and proves, among others, the following results: If f and g belong to such a semi-algebra F, then $\max(f, g)$ and $\min(f, g)$ are also in F, and there is a partial ordering in E such that F is the class of all monotonic increasing (i.e., $f(s) \leqq f(s')$ for $s \leqq s'$) non-negative continuous functions on E. The special case where E is a compact topological group is also discussed. The proof is based upon certain generalizations of Kakutani's theorem on $C(E)$ in which linear subspaces of $C(E)$ are replaced by convex subsets of $C(E)$. *K. Iwasawa* (Cambridge, Mass.)

2893:

Sebastião e Silva, J. Origin of the theory of distributions and its relation to physics and engineering. Ciência. Lisboa No. 15/16 (1958/59), 5–27. (Portuguese)

Expository.

2894:

Honda, Kôji; Yamamuro, Sadayuki. A characteristic property of L_p-spaces ($p>1$). Proc. Japan Acad. **35** (1959), 446–448.

The authors gave another characterization of the general L_p spaces from the standpoint of modular spaces. Using the terminologies in the book by H. Nakano, *Modulared semi-ordered linear spaces*, Maruzen, Tokyo, 1950 [MR **12**, 420], if a Banach lattice R is at least two dimensional and semi-regular, and there exists a conjugately similar correspondence T from R onto the conjugate space $\bar{R}$ such that

$$(Tx, x) = \|Tx\|\,\|x\| \quad (0 \leqq x \in R),$$

then R can be represented as a L_p space for some $p>0$. *H. Nakano* (Kingston, Ont.)

2895:

Hill, C. K. Corrigendum: The Hilbert bound of a certain doubly-infinite matrix. J. London Math. Soc. **35** (1960), 128.

The author corrects two errors in a previous paper [same J. **32** (1957), 7–17; MR **18**, 812]. (I) In the enunciation of theorem 5 (iii) the last inequality $\leqq$ should be $<$. (II) The proof of theorem 5 (ii) is not valid when $-\frac{1}{2} \leqq \lambda < 0$ because $l_n < 0$. The result is, however, correct, and the author completes the proof. *A. E. Ingham* (Cambridge, England)

2896:

Gribanov, Yu. I. Complete continuity of matrix operators in vector spaces. Dokl. Akad. Nauk SSSR **129** (1959), 975–978. (Russian)

Let l be a Banach space of sequences $X=\{x_n\}$ of real numbers and let $\tilde{l}$ denote the complementary space consisting of all sequences $Y=\{y_n\}$ such that $\sum_n x_n y_n$ converges for every $X \in l$. It is assumed that the norm on l is given by $\|X\| = \sup_{Y\in S} |\sum_n x_n y_n|$ for some subset S of $\tilde{l}$. Let $A=(a_{mn})$ be a matrix defining a bounded operator of l into itself and let E_n denote the matrix with all 0 entries except for 1's in the first n positions along the main diagonal. Author considers relations between the compactness of A and the conditions (α) $\|A - E_n A\| \to 0$, (β) $\|A - AE_n\| \to 0$ and (ω) $\|A - E_n A E_n\| \to 0$. Main results: (1) (α) and (β) are dual with respect to transposition of A; both imply compactness and together are equivalent with (ω). (2) If E_n tends strongly to the identity on l [$\tilde{l}$] then every compact A satisfies (α) [(β)]. Applications to l_p spaces are indicated. No proofs are given. {Some relevant

assumptions concerning l would seem to have been omitted (though the theorems announced are certainly valid in the case $l=l_p$).} *A. Brown* (Houston, Tex.)

2897:

Rota, G. C. Note on the invariant subspaces of linear operators. Rend. Circ. Mat. Palermo (2) 8 (1959), 182–184.

Let H be a separable hilbert space. In the hilbert space H' of square-summable sequences $(x_1, x_2, \cdots)$ $(x_k \in H)$, let U be the 'unilateral shift' operator defined by $U(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$. Theorem: Given any operator T on H with $\|T\| < 1$, there exists M, a closed invariant subspace of U, such that T is similar to the restriction of U to M. In the author's terminology, U is a 'universal model' for bounded operators. {Not cited is M. Schreiber's theorem [Duke Math. J. **23** (1956), 579–594; MR **18**, 748], which differs in that U is the unitary 'bilateral shift' operator; similarity is replaced by unitary equivalence; and a compression of U (to a non-invariant subspace) is used.} The proof is surprisingly simple. As the author says, "the result is easily extended to more general linear spaces".

C. Davis (Providence, R.I.)

2898:

Rota, Gian-Carlo. On models for linear operators. Comm. Pure Appl. Math. **13** (1960), 469–472.

In the author's theorem quoted in the preceding review, the hypothesis $\|T\| < 1$ is weakened to $\lim \|T^n\|^{1/n} < 1$, with hardly any alteration in the proof. An application is a short proof of a theorem of Sz.-Nagy [Magyar Tud. Akad. Mat. Kutató Int. Közl. **4** (1959), 89–93; MR **21** #7436]. *C. Davis* (Providence, R.I.)

2899:

Rota, Gian Carlo. On the representation of averaging operators. Rend. Sem. Mat. Univ. Padova **30** (1960), 52–64.

Like Moy [Pacific J. Math. **4** (1954), 47–63; MR **15**, 722], the author gives a characterization of those operators A on functions over a probability space which are conditional expectations. The author's conditions are these (p fixed $\in [1, \infty)$): (1) A is a contraction on L_p; (2) for $f \in L_p$ and g essentially bounded, $A(gAf) = (Ag)Af \in L_p$ ('averaging identity'); (3) $AI = I$, where I is identically $=1$. The same conditions for $p=\infty$ are satisfied by a wider class of operators, which the author describes. He also proves that his conditions for $p=\infty$ do characterize the conditional expectations if this condition is adjoined: $\int Af = \int f$ for all f. *C. Davis* (Providence, R.I.)

2900:

Davis, Chandler. Various averaging operations onto subalgebras. Illinois J. Math. **3** (1959), 538–553.

The paper studies and classifies averaging operations in finite-dimensional von Neumann algebras $\mathfrak{A}$ (faithfully representable as direct sums of full matrix algebras over the complex field). An 'averaging operation' $\mathscr{C}$ of $\mathfrak{A}$ onto R is a positive ($\mathscr{C}A$ is positive definite if $A \in \mathfrak{A}$ is positive definite, that is, $\mathscr{C}A \geq 0$ if $A \geq 0$) idempotent ($\mathscr{C}(\mathscr{C}A) = \mathscr{C}A$) operator mapping $\mathfrak{A}$ onto a self-adjoint subalgebra $\mathscr{R}$, which satisfies the identity (the so-called 'averaging identity') $\mathscr{C}(A\mathscr{C}B) = (\mathscr{C}A)(\mathscr{C}B)$. The averaging identity has been introduced by J. Kampé de Fériet [La Science Aérienne **3** (1934), 9–34; **4** (1935), 12–52] and studied by G. Birkhoff [Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 143–153; MR **13**, 361], S. T. C. Moy [Pacific J. Math. **4** (1954), 47–63; MR **15**, 722] and others in the case of commutative algebras.

(1) Suppose $\mathfrak{A}$ is a full matrix algebra, and $\mathscr{R} \geq \mathscr{R}'$, where $\mathscr{R}'$ is the commutor of $\mathscr{R}$ in $\mathfrak{A}$ ($\mathscr{R}'$ = all operators of $\mathfrak{A}$ commuting with each operator of $\mathscr{R}$). Then $\mathscr{R}$ is isomorphic to a direct sum of subalgebras isomorphic to full matrix subalgebras of $\mathfrak{A}$. It is shown that there is then exactly one averaging operation (called a 'pinching') of $\mathfrak{A}$ onto $\mathscr{R}$, which can be characterized in any of the following ways: (a) $\mathscr{C}A = \sum_i P_i A P_i$, where the P_i are all the minimal projections of $\mathscr{R}'$; (b) if $\|B\|_2 = (\mathrm{tr}\, B^*B)^{1/2}$, then $\mathscr{C}A$ is that B in $\mathscr{R}$ for which $\|B-A\|_2$ is a minimum; (c) $\mathscr{C}A$ is the average of XAX when X ranges over all symmetries in $\mathscr{R}'$; (d) $\mathscr{C}A$ is the average of U^*AU when U ranges over all unitary matrices in $\mathscr{R}'$.

(2) When $\mathscr{R}$ is a general selfadjoint subalgebra of $\mathfrak{A}$, the general averaging operation $\mathscr{C}$ onto $\mathscr{R}$ can be expressed in terms of a pinching, followed by a convex linear combination of homomorphisms, as follows. A selfadjoint subalgebra $\mathscr{B}$ can be found such that $\mathscr{B} \leq \mathfrak{A}$, $\mathscr{B} \geq \mathscr{R}$, $\mathscr{B} \geq \mathscr{B}'$ and $\mathscr{B}' = \mathscr{R}' \cap \mathscr{B}$. If $\mathscr{P}$ is the pinching from $\mathfrak{A}$ onto $\mathscr{B}$, it is shown that $\mathscr{C}\mathscr{P} = \mathscr{C}$, and that, if $M(\mathscr{B}, \mathscr{R})$ is the (convex) set of averaging operations from $\mathscr{B}$ to $\mathscr{R}$, the extreme points of $M(\mathscr{B}, \mathscr{R})$ are exactly those averaging operations which are also homomorphisms. It is also shown that such a subalgebra $\mathscr{B}$ need not be unique.

(3) Let A be fixed hermitian operator. The relationship between the spectrum of A and that of $\mathscr{C}A$, for a varying averaging operation $\mathscr{C}$, is studied. The following are the main results: (a) If $\alpha = (\alpha_1, \cdots, \alpha_n)$ are the eigenvalues of A, then $\beta = (\beta_1, \cdots, \beta_n)$ are the eigenvalues of $\mathscr{C}A$ for some pinching $\mathscr{C}$ if and only if $\beta = S\alpha$ for some doubly stochastic $n \times n$ matrix S; the characterization of β when $\mathscr{C}$ may be any averaging, is also given. (b) If $K(A)$ is the convex hull of the operators unitarily equivalent to A and $B(A)$ the set of all pinchings of operators unitarily equivalent to A, then $B(A) = K(A)$. (c) f is a Schur-convex function if and only if $f(\mathscr{C}A) \leq f(A)$ for all selfadjoint A and pinchings $\mathscr{C}$; the characterization of f when $\mathscr{C}$ may be any averaging, is also given (for dimension ≥ 4).

G.-C. Rota (Cambridge, Mass.)

2901:

Tarnopol's'kiĭ, V. G. On the self-adjointness of difference operators with operator coefficients. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 1189–1192. (Ukrainian. Russian and English summaries)

Author's summary: "In this note the author considers a difference operator which results from the difference expression

$$L[u]_j = A_{j-1}u_{j-1} + B_j u_j + A_j u_{j+1},$$

where u_j is a sequence of elements of Hilbert's space H and A_j, B_j are bounded self-adjoint operators in H. The author obtains some sufficient conditions of the self-adjointness of the difference operator. These theorems are generalizations of the corresponding theorems of the classical moment problem, of the matrix moment problem and partially of the theory of equations in partial differences."

2902:

Gol'dman, M. A.; Kračkovskiĭ, S. N. On operators of Riesz-Schauder type. Uspehi Mat. Nauk **14** (1959), no. 6 (90), 159–164. (Russian)

An operator T (between Banach spaces) is of Riesz-Schauder type if the equations $Tx=y$ and $T^*\phi=f$ are both normally solvable (i.e., T and its adjoint T^* have closed resp. weakly closed range) and if T and T^* possess null spaces of equal finite dimension. If T is closed and bounded then elementary arguments show that necessary and sufficient conditions that T be of Riesz-Schauder type are that T have the form $T=U+K$ or equivalently the form $T=U+V$, where K denotes an operator of finite rank, V a compact operator and U a bounded closed operator with bounded inverse. If T is merely required to be closed the theorem remains true if U is allowed to be the (not necessarily bounded) inverse of a bounded closed operator. Author gives a new proof of this fact based on the following lemma: If U is as stated and if M_U denotes the set of those compact operators V for which $U+V$ is of Riesz-Schauder type, then M_U is both open and closed in the set of all compact operators.

A. Brown (Houston, Tex.)

2903:

Heuser, Harro. On the spectral theory of symmetric finite operators. Trans. Amer. Math. Soc. **94** (1960), 327–336.

Let A be a symmetric, bounded, positive operator on a not necessarily complete Hilbert space, A being "finite" in the sense that, for each $\lambda\neq 0$, the dimension of the kernel of $A-\lambda I$ and the codimension of the range of $A-\lambda I$ are finite and equal. Starting from two results about symmetric finite operators, one due to the author [*Über Operatoren mit endlichen Defekten*, Dissertation, Tübingen, 1956; Math. Z. **74** (1960), 165–185; Satz. 6] and the other to Wielandt [Math. Z., loc. cit., Satz. 9], this paper examines an iteration process for locating characteristic values of A and also the structure of the spectrum of A. A number $\lambda\neq 0$ is said to contribute to an element x of X if there exists $e\neq 0$ in X such that $Ae=\lambda e$ and $(x,e)\neq 0$. Several theorems are given relating the behaviour of the sequence A^kx $(k=1, 2, \cdots)$ with the set of numbers contributing to x. For example, if $Ax\neq 0$, $\lim_{k\to\infty}(\|A^kx\|/\|A^{k-1}x\|)$ exists and equals the supremum μ_x of all characteristic values of A contributing to x. Various necessary and sufficient conditions are given in order that this μ_x shall itself contribute to x. The numbers μ_x are involved in Wavre's concept of regularity of A [Comment. Math. Helv. **15** (1943), 299–317; MR **5**, 272], which is here expressed in terms of the sets of characteristic values of A contributing to elements x such that $Ax\neq 0$. Regularity of A^* is expressed in terms of the spectrum of A. Finally, the restriction $A\geq 0$ is shown to be removable by applying the preceding results to A^2.

R. E. Edwards (Reading)

2904:

Strang, W. G. On the Kantorovich inequality. Proc. Amer. Math. Soc. **11** (1960), 468.

The author proves the following generalization of a version of Kantorovich's inequality given by W. Greub and this reviewer [same Proc. **10** (1959), 407–415; MR **21** #3774]. If T is a linear operator on a Hilbert-space H and $\|T\|=M$, $\|T^{-1}\|=m^{-1}$, then for all $x, y\in H$

$$|(Tx,x)(y,T^{-1}y)| \leq \frac{(M+m)^2}{4mM}(x,x)(y,y).$$

The bound is best possible.

W. C. Rheinboldt (Syracuse, N.Y.)

2905:

Kužel', A. V. Spectral analysis of unbounded non-selfadjoint operators. Dokl. Akad. Nauk SSSR **125** (1959), 35–37. (Russian)

In a previous article [Dokl. Akad. Nauk SSSR **119** (1958), 868–871; MR **20** #6091], the author defined K^r-operators (non-Hermitian operators A with deficiency index (r,r) such that dim $D_A=r$ (mod D_{A_0}) where A_0 is the Hermitian restriction of A) and considered the case $r=1$. In the present note, analogous theorems are given for $1\leq r<\infty$ using some results of V. P. Potapov [Trudy Moskov. Mat. Obšč. **4** (1955), 125–236; MR **17**, 958].

M. Katětov (Prague)

2906:

Cordes, H. O. A matrix inequality. Proc. Amer. Math. Soc. **11** (1960), 206–210.

A new elementary proof is given to an inequality by Heinz [Math. Ann. **123** (1951), 415–438; MR **13**, 471] to the effect that $|(Qu,v)|\leq\|A^ru\|\,\|B^{1-r}v\|$ if $\|Qu\|\leq\|Au\|$, $u\in D[A]$, and $\|Q^*u\|\leq\|Bu\|$, $u\in D[B]$ (A, B, Q are linear operators in a Hilbert space, A, B self-adjoint positive, Q closed). The proof is based on the lemma that if (u,v) and $(u,v)_0$ are two inner products in a vector space with the mutually equivalent norms $\|u\|$, $\|u\|_0$, and if T is Hermitian symmetric with respect to $(\ ,\ )_0$, then $\|T\|_0\leq\|T\|$. Other proofs have been given by the reviewer [ibid. **125** (1952), 208–212; MR **14**, 766], Dixmier [ibid. **126** (1953), 75–78; MR **15**, 39] and Heinz [Report of an international conference on operator theory and group representations, Arden House, Harriman, N.Y., 1955, pp. 27–29, National Academy of Sciences—National Research Council, Washington, D.C., 1955; MR **18**, 35].

T. Kato (Tokyo)

2907:

Mihlin, S. G. On the convergence of a direct method. Uspehi Mat. Nauk **15** (1960), no. 1 (91), 221–223. (Russian)

For solving $Lu\equiv Au+Bu=f$, where A and B are linear operators in a separable Hilbert space H, the method under consideration is to approximate the solution u_0 by $u_n=\sum_1^n a_k\phi_k$, where the a_k satisfy

$$(Lu_n, A\phi_k) = (f, A\phi_k).$$

In two theorems conditions are obtained assuring the convergence $u_n\to u_0$, $Lu_n\to f$. To summarize only the first theorem, the conditions are: (1) the solution u_0 exists uniquely; (2) A^{-1} is bounded and defined throughout H; (3) BA^{-1} is completely continuous in H; (4) the system $\{A\phi_k\}$ is complete in H.

A. S. Householder (Oak Ridge, Tenn.)

2908:

Edwards, R. E. The stability of weighted Lebesgue spaces. Trans. Amer. Math. Soc. **93** (1959), 369–394.

Let X denote a locally compact space, μ a non-zero positive Radon measure on X, and S a semigroup of homeomorphisms of X onto itself containing the identity map. Suppose that S admits a real-valued function Δ satisfying $\mu(s^{-1}(E)) = \Delta(s)\mu(E)$ for $s \in S$ and μ-integrable sets E. Also let w be a fixed positive, finite-valued, locally μ-integrable "weight function" on X. For $1 \leq p < \infty$, let $L^p(w)$ [$\bar{L}^p(w)$] denote all finite real-valued μ-measurable functions f on X such that $|f|^p w$ is μ-integrable [essentially μ-integrable]. The author studies the stability of these spaces under S; e.g., when does $f \in \bar{L}^p(w)$ imply $f \circ s \in \bar{L}^p(w)$ for all $s \in S$?

The function w is moderate if there exists a function M on S such that $\int |f \circ s| w d\mu \leq M(s) \int |f| w d\mu$ for $f \in \bar{L}^1(w)$ and $s \in S$. Theorems: $\bar{L}^p(w)$ is stable if and only if $\bar{L}^1(w)$ is stable. If w is moderate, then $\bar{L}^1(w)$ is stable. If $w > 0$ holds locally almost everywhere and $\bar{L}^1(w)$ is stable, then w is moderate. If μ is diffuse (continuous) and if, whenever A and B are bounded μ-integrable sets such that $\mu(A) > 0$ and $\mu(B) > 0$, there exists an $s \in S$ such that $\mu(A \cap s^{-1}(B)) > 0$, then $\bar{L}^1(w)$ is stable if and only if w is moderate. A similar result holds for atomic measures μ.

These results are applied to the case that X is a topological group, μ is left Haar measure, and S is all left (or right) translations of X. For example, if X is compact and infinite, then $L^p(w)$ is stable under left translations if and only if $w = 0$ a.e. or else w and w^{-1} are essentially bounded. The author also studies $\bar{L}^1(w)$ as a convolution algebra.

K. A. Ross (Seattle, Wash.)

2909:

de Leeuw, Karel. On the adjoint semigroup and some problems in the theory of approximation. Math. Z. 73 (1960), 219–234.

P. L. Butzer [Math. Ann. 133 (1957), 410–425; MR 20 #1232] has studied the saturation problem in approximation theory using the theory of semigroups of operators on a reflexive Banach space. In this paper the underlying method is ingeniously extended to Banach spaces which need not be reflexive, and then applied to a variety of specific problems. A typical result is the following. For each $t > 0$ let T_t be the mapping of $L^1(-\infty, \infty)$ into itself defined by

$$(T_t f)(x) = (\pi t)^{-1/2} \int_{-\infty}^{\infty} f(x-y) \exp(-y^2/t) dy.$$

Then the following statements are equivalent: (i) $\|T_t f - f\| = O(t)$ as $t \to 0$; (ii) f'' is a measure. Furthermore if $\|T_t f - f\| = o(t)$ as $t \to 0$ then $f \equiv 0$. The corresponding result for $L^p(-\infty, \infty)$, $1 < p < \infty$, was previously obtained by Butzer in the paper mentioned above. The case $p = \infty$ is also dealt with in the present paper.

I. I. Hirschman, Jr. (Huntington, W. Va.)

2910:

Rabson, Gustave. The existence of nonabsolutely convergent Fourier series on compact groups. Proc. Amer. Math. Soc. 10 (1959), 893–897.

It is shown how to construct explicitly on a compact group of positive dimension a continuous function which is not an absolutely convergent Fourier transform. The proof involves extension from the case of a circle group to that of a compact Lie group, followed by approximation by such groups of a general group of the type described.

I. E. Segal (Cambridge, Mass.)

2911:

Kesten, Harry. Full Banach mean values on countable groups. Math. Scand. 7 (1959), 146–156.

On a countable group $\mathfrak{G}$ a function $p(x)$ $(x \in \mathfrak{G})$ is considered which satisfies $p(x) \geq 0$, $p(x) = p(x^{-1})$, $\sum_{x \in \mathfrak{G}} p(x) = 1$, and such that $\mathfrak{G}$ is generated by the set $\{x | p(x) > 0\}$. It is shown that an invariant mean on the set of all bounded functions on $\mathfrak{G}$ exists if and only if the stochastic matrix $\|m_{x_i, x_j}\|$ with $m_{x_i, x_j} = p(x_i^{-1} x_j)$ has spectral radius 1. The proof is based on a theorem of the reviewer [Math. Scand. 3 (1955), 243–254; MR 18, 51].

E. Følner (Copenhagen)

2912:

Gel'fand, I. M.; Pyateckiĭ-Šapiro, I. I. Theory of representations and theory of automorphic functions. Uspehi Mat. Nauk 14 (1959), no. 2 (86), 171–194. (Russian)

Let G denote a semi-simple Lie-group with a discrete subgroup Γ such that the left coset space $X = G/\Gamma$ is compact. Let K denote the space of square-integrable functions on X. The authors prove that the space K can be expressed as a denumerable direct sum of primitive representations of G and each representation occurs with finite multiplicity. The proof is based on the Hilbert-Schmidt theory of Fredholm equations.

Let Ω be a linear space in which there exists a continuous irreducible representation of G. It is possible to define "matrix elements" $f_{\xi,\eta}(g)$, $\xi, \eta \in \Omega$, $g \in G$ such that $f_{\xi,\eta}(g)$ for given η is defined for some values of ξ, and such that $f_{\xi,\eta}(g) = f_{\eta,\xi}(g^{-1})$, while $f_{\xi,\eta}(g)$ is linear in ξ and η, and is connected with the representation $T_g \xi$ by the relation

$$f_{T_{g_1}\xi, T_{g_2}\eta}(g) = f_{\xi,\eta}(g_2^{-1} g g_1).$$

Further $f_{\xi,\eta}(g)$ is continuous in (ξ, η). The space with the matrix elements is called "full" if the convergence of a sequence $f_{\xi,\eta_n}(g)$ implies the convergence of η_n. Every Hilbert space in which G admits an irreducible representation, can be imbedded in a "full" space, and the dimension of this space will be equal to the multiplicity with which the irreducible representation occurs in the above-mentioned expression for K.

The results are interpreted as results on general automorphic functions generalizing the well-known theories of Siegel and Selberg.

H. Tornehave (Copenhagen)

2913:

Sakai, Shôichiro. On a conjecture of Kaplansky. Tôhoku Math. J. (2) 12 (1960), 31–33.

The author proves Kaplansky's conjecture that any derivation of a C^*-algebra is automatically continuous.

C. R. Putnam (Lafayette, Ind.)

2914:

Edwards, R. E. States of operator algebras. Studia Math. 19 (1960), 63–75.

This treats the representability of a positive linear functional f on a self-adjoint algebra of bounded linear operators on a Hilbert space H, in the form

$$f(T) = \int_{\Sigma} (Tx, x) dm(x),$$

where x ranges over the unit ball Σ of H taken in its weak topology, and m is a Radon (=regular) measure on

Σ. A typical result is that if $f(T) \leq \sup_{x \in \Sigma} (Tx, x)$, for all self-adjoint T in the algebra A, then the foregoing representation holds for all compact (= completely continuous) T in A, and some measure m with $m(\Sigma) \leq 1$.

I. E. Segal (Cambridge, Mass.)

2915:

Glimm, James G. On a certain class of operator algebras. Trans. Amer. Math. Soc. **95** (1960), 318–340.

The author studies C^*-algebras $\mathfrak{A}$ which are uniform closures of ascending sequences $\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \cdots$ of subalgebras, each $\mathfrak{A}_k$ being a factor of type $\mathrm{I}_{n(k)}$, $n(k) < \infty$. Such an $\mathfrak{A}$ is called uniformly hyperfinite (UHF). First, the UHF algebras are classified * algebraically (theorem 1.12): if also $\mathfrak{B}$ is UHF, obtained from $\mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \cdots$, where $\mathfrak{B}_k$ is a factor of type $\mathrm{I}_{m(k)}$, then $\mathfrak{A}$ is *-isomorphic to $\mathfrak{B}$ if and only if, for each k, there exists k' such that $m(k)$ divides $n(k')$ and $n(k)$ divides $m(k')$. Another characterization is given (theorem 1.13) of the class of UHF algebras and finite type I factors: (a) for each $A_1, \cdots, A_n$ in the C^*-algebra $\mathfrak{A}$, and each $\varepsilon > 0$, there exists some type I finite subfactor $\mathfrak{B}$ such that each A_i lies within ε of $\mathfrak{B}$; and (b) $\mathfrak{A}$ is separable in the uniform topology.

The pure states of $\mathfrak{A}$ are studied. For example (theorem 2.8) they are w^* dense in all states of $\mathfrak{A}$. Concerning the irreducible representations ϕ_ρ and ϕ_τ obtained from the pure states ρ and τ, the following facts are shown equivalent, in theorem 3.4: (a) ϕ_ρ is unitarily equivalent to ϕ_τ; (b) there exists U unitary in $\mathfrak{A}$ with $\rho(U^*AU) = \tau(A)$ for all $A \in \mathfrak{A}$. These are also related to certain equivalences within the approximating $\mathfrak{A}_k$. Theorem 3.5: The Hilbert space of any irreducible representation of a UHF algebra is separable. Theorem 4.1: If ϕ is a representation of a UHF algebra $\mathfrak{A}$, and $\phi(\mathfrak{A})$ has a weak closure of finite type, then this weak closure is an approximately finite factor.

J. Feldman (Berkeley, Calif.)

2916:

Shimoda, Isae. Notes on general analysis. VIII. J. Gakugei Tokushima Univ. **10** (1959), 29–31.

[For part VII see same J. **9** (1958), 19–20; MR **21** #3782.]

This note deals with a one-to-one analytic mapping f of the open unit disk in the complex plane onto a neighborhood D_f of 0 in a complex Banach space X, with analytic inverse f^{-1}, subject to the condition $\|f'(0)\| = 1$, $f(0) = 0$. The Banach space is then necessarily one-dimensional. The substance of the note is that there is a constant $\delta > 0$ such that for every such f the image of the unit disk contains the sphere $\{x : \|x\| < \delta\}$. {It seems to the reviewer that this is an immediate consequence of a theorem of this kind in classical function-theory [see, for example, section 6.8 in Titchmarsh, *Theory of functions*, Clarendon, Oxford, 1932], together with the fact that X is essentially the complex plane with an equivalent topology given by a Minkowski norm.}

The reviewer takes this occasion to correct a typographical error in the review of part VII of this series of papers [see reference above]. In the last sentence of the review, for X read E_1. *A. E. Taylor* (Los Angeles, Calif.)

2917:

Schaefer, Helmut. On the singularities of an analytic function with values in a Banach space. Arch. Math. **11** (1960), 40–43.

A cone in a real or complex vector space E is a subset K of E such that $K + K \subset K$ and $\lambda K \subset K$ for $\lambda > 0$. If E is a normed space a cone K is called normal if there is a norm $\|\cdot\|$ equivalent to the given one such that $\|x+y\| \geq \|y\|$ for x and y in K. Now let E be a Banach space and let K be a normal cone in E, then the main theorem states that, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 1, where z is a complex number and $a_n \in K$, then $z = 1$ is a singular point of f. If in addition $z = 1$ is a pole of order k, then every other pole of f on $|z| = 1$ is of order $\leq k$. The application of the main theorem to the complex plane yields a theorem which is related to a theorem of Pringsheim, but neither of them is a special case of the other. For a normal cone in the complex plane is a sector with vertex at 0 and of central angle $< \pi$. Another consequence of the main theorem: If A is a complex Banach algebra and if $a \in A$ has the property that the cone spanned by the set $\{a^n : n \geq 0\}$ is normal, then the spectral radius of a is in the spectrum of a. *I. Namioka* (Ithaca, N.Y.)

2918:

Calyuk, Z. B. A remark concerning the application of solubility conditions in Čaplygin's problem to problems of the qualitative theory of equations. Dokl. Akad. Nauk SSSR **134** (1960), 52–54 (Russian); translated as Soviet Math. Dokl. **1**, 1028–1030.

Let P_1 and P_2 be mappings of the sets N_1 and N_2 respectively into themselves and y_i such that $y_i = P_i(y_i)$. Let T be an operator mapping the product $N_1 \times N_2$ into a partially ordered set R. Let F be a mapping of R into R having property (M) that for each y such that $y = F(y)$ the inequality $z \leq F(z)$ $[z \geq F(z)]$ implies $z \leq y$ $[z \geq y]$. Suppose finally that $T(y_1, P_2(x_2)) \leq F(T(y_1, x_2))$, $x_2 \in N_2$. Then $T(y_1, y_2) \leq y$ for any y satisfying $y = F(y)$. The author formulates this lemma and indicates a wide range of applications to questions involving integral equations of Volterra type, implications of solubility, asymptotic stability, etc. *A. N. Milgram* (Minneapolis, Minn.)

CALCULUS OF VARIATIONS

2919:

Cinquini, Silvio. Sopra l'esistenza dell'estremo per una classe di integrali curvilinei in forma parametrica. Ann. Mat. Pura Appl. (4) **49** (1960), 25–71.

This paper continues the author's study of variational integrals of the type

$$\int_C \Phi(x, y, z; x', y', z'; \cdots; x^{(n)}, y^{(n)}, z^{(n)})dt,$$

for $n = 2$ or 3 [see S. Cinquini, Rend. Circ. Mat. Palermo (2) **6** (1957), 271–288; MR **20** #4791]. It is devoted to the proof of basic semicontinuity and existence theorems.

W. H. Fleming (Providence, R.I.)

2920:

Lawden, D. F. Discontinuous solutions of variational problems. J. Austral. Math. Soc. **1** (1959/61), part 1, 27–37.

Dans le problème le plus élémentaire du Calcul des Variations, relatif à $I = \int_a^b f(x, y, y')dx$, on appelle quelquefois discontinues les extrémales (que l'on devrait

plutôt appeler anguleuses) pour lesquelles y' est discontinue. C'est des extrémales discontinues pour lesquelles y lui-même est discontinue que s'occupe ici D. F. Lawden. Il commence par préciser le sens qu'il attribue à I pour y discontinue en c, en substituant à y une fonction continue qui n'en diffère que dans $(c-\delta, c+\delta)$ et qui est linéaire dans cet intervalle; si on suppose alors que f/y' tend vers $k(x, y)$ uniformément dans $(c-\xi, c+\xi)$, $(y(c-0)-\eta, y(c+0)+\eta)$, il montre que, pour $\delta \to +0$, $I(\delta) \to \int_a^{c-0} f dx + \int_{c+0}^b f dx + \int_{y(c-0)}^{y(c+0)} k(c, u) du$. Sous les hypothèses précédentes, il montre que $\int_a^b f(x, y, y')dx$ est stationnaire relativement aux variations faibles de y (satisfaisant à $y(a)=A$, $y(b)=B$) pourvu que (1) là où y et sa dérivée sont continues, $f_y - d(f_{y'})/dx = 0$ (Euler); (2) là où y' est discontinue, les conditions de Weierstrass-Erdmann sont satisfaites; (3) au point c où y est discontinue, $f - y'f_{y'} - \int k_c dy$ est continue, et $f_{y'} = k$ des deux côtés du point de discontinuité. Il applique ce résultat à l'exemple $\int_0^1 (y'^3 - 1)^{1/3} dx$ avec $y(0)=0$, $y(1)=2$: le minimum absolu est réalisé par une extrémale discontinue formée de deux segments rectilignes de pente $2^{-1/3}$. Dans un dernier paragraphe, il montre comment on peut généraliser (1) au cas d'une intégrale faisant intervenir des dérivées d'ordre supérieur au premier, (2) au cas d'une intégrale faisant intervenir plusieurs fonctions inconnues.

{Il est intéressant de remarquer que c'est un problème physique: "rocket tracks of minimum propellant expenditure in a given gravitational field" qui a conduit l'auteur à examiner les problèmes du type précédent. Peut-être peut-on aussi rappeler à cette occasion les recherches différentes, déjà anciennes, de Rasmadze [Proc. International Math. Congress held in Toronto, August 11–16, 1924, Univ. of Toronto Press, Toronto; Vol. I, pp. 561–588.].}

M. Janet (Paris)

2921:

Ewing, George M. Lipschitzian parameterizations and existence of minima in the calculus of variations. Proc. Amer. Math. Soc. **11** (1960), 87–89.

The Weierstrass integral $W[x; a, b, f]$ of a function $f(x, r)$ ($x \in A$, $r \in E_n$, A a closed subset of E_n) on any continuous rectifiable curve $x = x(t)$ ($a \leqq t \leqq b$, $x(t) \in A$)—briefly, a map x—exists under the usual conditions: (I) $f(x, r)$ continuous in $A \times E_n$; (II) $f(x, kr) = kf(x, r)$, $k \geqq 0$, $x \in A$, $r \in E_n$. Then (II) assures that W depends only on the Fréchet equivalence class of the map x. For f satisfying (I), (II), and (III) $f(x, r) > 0$, $x \in A$, $r \in E_n$, $r \neq 0$, a map x is said to be f-Lipschitzian (briefly, fL) if (IV) $f[x(\tau), x(t') - x(t)] \leqq W[x; t, t', f] + |x(t) - x(t')|^2$ for all $a \leqq t \leqq \tau \leqq t' \leqq b$. Then x is fL if and only if x is Lipschitzian. An existence proof is given of the minimum in calculus of variations: If (I), (II), (III) are satisfied, if X is the class of all maps x satisfying (IV), whose graph joins disjoint closed subsets of A, and X is not empty, then there exists $x_0 \in X$ minimizing W in X. Here f is not supposed to be convex.

L. Cesari (Ann Arbor, Mich.)

GEOMETRY

2922:

Strommer, J. Ein einfaches Beispiel für die Unabhängigkeit des Hilbertschen Axioms III 5. Acta Math. Acad. Sci. Hungar. **10** (1959), 395–396. (Russian summary, unbound insert)

Hilbert's axiom III 5 (which establishes the interrelation of congruent angles and segments) is proven to be independent of all the other axioms. For this purpose the author constructs a model in which these and not III 5 holds, by defining congruence of segments so that even the first theorem on congruent triangles (a consequence of III 5) fails to be always fulfilled. [D. Hilbert, *Grundlagen der Geometrie*, 8th ed., Teubner, Stuttgart, 1956; MR **18**, 227.]

S. R. Struik (Cambridge, Mass.)

2923:

Lingenberg, Rolf. Zur Kennzeichnung der Gruppe der gebrochen-linearen Transformationen über einem Körper von Charakteristik $\neq 2$. Arch. Math. **10** (1959), 344–347.

Es sei G eine "zweispiegelige" Gruppe, d.h. eine Gruppe, in der jedes Element als Produkt von zwei involutorischen Elementen darstellbar ist; ferner gelte in der Menge J der involutorischen Elemente von G das "Transitivitätsgesetz": Ist $a \neq b$ und sind abc, abd involutorisch, so ist acd involutorisch ($a, b, c, d \in J$). In J lassen sich dann in bekannter Weise Büschel bilden, d.h. maximale Teilmengen mit der Eigenschaft, dass das Produkt von je drei Elementen einer Teilmenge involutorisch ist [vgl. das Buch des Ref.: *Aufbau der Geometrie aus dem Spiegelungsbegriff*, Springer, Berlin, 1959; MR **21** #6557; S. 132f.]. Es gibt Büschel, welche zwei Elemente mit involutorischem Produkt enthalten; Büschel, die nicht zwei solche Elemente enthalten, werden Enden genannt. Verf. beweist: Wenn je zwei Enden ein Element gemein haben, gehört kein involutorisches Element mehr als zwei Enden an. Als Beweismittel dient das Bergausche Lemma vom Ende [loc. cit., S. 222], das, geeignet formuliert, unter den vorliegenden Voraussetzungen als gültig erwiesen wird.

Ref. hat die Gruppen $PGL_2(K)$ über den Körpern von Charakteristik $\neq 2$ als abstrakte zweispiegelige Gruppen durch das Transitivitätsgesetz und drei weitere Gesetze über die involutorischen Elemente gekennzeichnet [loc. cit., § 11]. Der Satz des Verf. lehrt, dass das letzte von diesen Gesetzen (UV2) aus den übrigen Forderungen folgt und daher in der Kennzeichnung entbehrt werden kann.

F. Bachmann (Kiel)

2924:

Yff, Peter. On the Brocard points of a triangle. Amer. Math. Monthly **67** (1960), 520–525.

A broken line meeting the sides of a triangle $A_1A_2A_3$ (counterclockwise) each at the same angle θ will have a series of points on A_2A_3. Conditions are given for θ (in terms of the Brocard angle ω) that these points converge. Moreover, except for $\theta = \omega$, there always exists a triangle (T_1), inscribed in a triangle (T), its sides clockwise meeting those of (T) at the same angle θ. If (T_2) is inscribed in (T_1) by the same rule, etc., the sequence of such inscribed triangles converges toward Ω_1, the first Brocard point of (T). If (T_1) has the vertices B_1, B_2, B_3 the envelope of B_1B_2 lines, as θ varies, is a parabola. The three parabolas thus obtained for the three sides of (T_1) have a common focus.

S. R. Struik (Cambridge, Mass.)

2925:

Tyrrell, J. A. On the enumerative geometry of triangles. Mathematika **6** (1959), 158–164.

A triangle, in the sense of Schubert [Math. Ann. **17**

(1880), 153–212], is an entity in the projective plane which consists of: (a) an ordered triad of points (P_1, P_2, P_3); (b) an ordered triad of lines (l_1, l_2, l_3) connected with the points P_i by the incidence relations $P_i \subset l_j$ $(i \neq j)$; and (c) a three base-point net Φ of conics with the P_i as base-points. This system of triangles is more extensive than the natural system defined by (a) and (b) only, and has been discussed by Semple [Mathematika **1** (1954), 80–88; MR **16**, 614]. These triangles can be coordinatised, and are represented on a non-singular algebraic variety of 6 dimensions, Ω_6. In the paper under review a basis for fivefolds on Ω_6 is obtained, using the method of degenerate collineations. Schubert's assumption, that fivefold systems of triangles are determined, for enumerative purposes, by the values of seven numerical characters, is confirmed. It will be recalled that the establishment of proper foundations for the Schubert calculus was one of Hilbert's Paris problems. The paper makes excellent use of modern rigorous methods. We may therefore certainly accept the concluding remark that the image-manifold has order 238,446.

D. Pedoe (Singapore)

2926:

Niče, Vilim. Die Büschel der Reyeschen tetraedralen Strahlenkomplexe und ihre projektive Zuordnung. Rad Jugoslav. Akad. Znan. Umjet. **314**. Odjel Mat. Fiz. Tehn. Nauke 7, 229–262 (1959). (Serbo-Croatian and German)

If α, β, γ, δ are the face-planes of a tetrahedron $ABCD$ and l is a line of general position, the four planes Al, Bl, Cl, Dl have the same cross ratio as the four points $\alpha \cdot l$, $\beta \cdot l$, $\gamma \cdot l$, $\delta \cdot l$. The locus of lines l for which this cross ratio has any particular value is a tetrahedral complex [T. Reye, *Die Geometrie der Lage*, Vol. 3, A. Kröner, Leipzig, 1910; pp. 5–6]. By allowing the value of the cross ratio to vary, the author obtains a "pencil" of tetrahedral complexes. Each line of general position belongs to one member of the pencil, but each line through a vertex or in a face-plane belongs to every member. He is particularly interested in a harmonic set of four such complexes. Finally, instead of a single tetrahedron he considers a set of two, three, or four tetrahedra of general relative position. He finds that the locus of lines on which the faces of the tetrahedra determine equal cross ratios is, for two tetrahedra, a quartic complex, for three, a congruence of order 12, and for four, a ruled surface of order 96.

H. S. M. Coxeter (Toronto)

2927:

Vincensini, Paul. Sur une représentation des cercles et des sphères de l'espace ordinaire sur l'espace euclidien à quatre dimensions. Boll. Un. Mat. Ital. (3) **14** (1959), 338–344. (Italian summary)

A sphere Σ of center ω and radius R in a euclidean space E_3 immersed in a euclidean space E_4 is represented by a point I of E_4 on the perpendicular to E_3 at ω and such that $\overline{\omega I}^2 + R^2 = \overline{IF}^2$, where F is a fixed point of E_4 outside of E_3. Each point I of E_4 is the image of a real or imaginary sphere of E_3; null spheres of E_3 are represented by the points of a hyperparaboloid P of E_4 with F as focus. A circle of E_3, considered as the base of a pencil Φ of spheres, is represented by a straight line of E_4, which is tangent to P if Φ consists of tangent spheres. One of the properties of this representation deals with the images of the circles orthogonal to an arbitrary surface S of E_3: they are the lines of the tangent planes to that surface S^1 of P which is orthogonally projected on E_3 in S.

D. J. Struik (Cambridge, Mass.)

2928:

Duschek, Adalbert; Hochrainer, August. ★Grundzüge der Tensorrechnung in analytischer Darstellung. In drei Teilen. I. Teil: Tensoralgebra. 4te ergänzte Aufl. Springer-Verlag, Vienna, 1960. viii + 171 pp. Paperbound: \$5.70.

The first part of this three volume "Tensorrechnung" appeared first in 1946 and is now, after the death of Professor Duschek, republished in a fourth edition by Professor Hochrainer. [For vols. II, III (1950, 1955), see MR **15**, 470; **17**, 214.] The present volume contains an exposition of the tensor algebra of three-dimensional euclidean space as an extension of ordinary vector algebra, all in index notation and in close relationship with the methods of common analytical geometry. The computational apparatus is therefore based on the use of the symmetric unit tensor δ_{ij} and the alternating unit tensor ε_{ijk}, with the relation $\varepsilon_{ijk}\varepsilon_{ihl} = \delta_{jh}\delta_{kl} - \delta_{kh}\delta_{jl}$ (the only formula worth knowing by heart). The subject is quite carefully explained and is not pursued beyond bivalent tensors; the final chapter deals with quadrics. A spirited introduction on what, according to the authors, tensors should and should not do, gives the reader a joyful spirit of anticipation, in which he probably will not be disappointed: the Duschek way of introducing tensors (somewhat like the way followed by McConnell in English) is still (or so this reviewer believes) the best for budding physicists and engineers, and may also be of use to the professional.—There are problems and solutions.

D. J. Struik (Cambridge, Mass.)

2929:

Coxeter, H. S. M. ★The real projective plane. 2nd. ed. Cambridge University Press, New York, 1960. xii + 226 pp. \$3.75.

Paperbound reprinting of the 1955 edition [MR **16**, 1143].

2930:

Supnick, Fred. A theorem concerning six points. Proc. Amer. Math. Soc. **11** (1960), 498–504.

Let a set S of 6 points P_i $(i = 1, \cdots, 6)$ be specified in a plane, no three collinear and not all on a conic. The interior of a conic is defined as the convex part of the plane bounded by the conic. Let C_i be the conic passing through all five P_j, $j \neq i$. Each S belongs to one of the following three types. (1) Each P_i is in the interior of its C_i. (2) Each P_i is in the exterior of its C_i. (3) One P_i is in the interior of C_i, and one P_j is in the exterior of C_j. The problem dealt with is which of these types are possible and under what conditions they occur.

The author uses the following definition. Out of 5 points, no 3 collinear, let either 5 or 3 lie on the boundary β of their convex hull. If β is a pentagon, denote its vertices by A, B, C, D, E cyclically. If β is a triangle, denote the interior points by D and E and the other three by A, B, C such that the half-lines $DE^{\rightarrow}$ and $ED^{\rightarrow}$ intersect the sides AB and BC of β, respectively. Then the intersection of the open angular regions CAD, DBE, ACE is called the nucleus of $ABCDE$.

Theorem: If b, the boundary of the convex hull of S, has 4 or 6 vertices, then S is of type (3). If b has 3 or 5 vertices, then S is of type (1) or (3) according as any element of S not on b belongs to or does not belong to the nucleus of the other five. Type (2) is impossible.

R. Artzy (Chapel Hill, N.C.)

2931:

Kárteszi, Ferenc. On projective mapping of quadrics to each other. Mat. Lapok **9** (1958), 260–272. (Hungarian)

In der vorliegenden Arbeit beschäftigt sich der Verf. mit solchen Kollineationen der Flächen zweiter Ordnung, bei welchen die Verbindungsgeraden der entsprechenden Punkte der Fläche und der Kegelfläche beide Flächen berühren. Im Laufe der Untersuchungen wird ein elementarer Beweis der Sätze von Bianchi und Nicoletti gegeben [s. L. Bianchi, Rend. Circ. Mat. Palermo **22** (1906), 75–96] welcher die rein geometrischen Methoden der reellen projektiven Geometrie benutzt, während die ursprünglichen Beweise auf koordinatengeometrischem Wege, unter Heranziehung des Körpers der komplexen Zahlen verliefen. Der Verf. untersucht auch die Bedingungen für die Realität der Kollineationen, und er weist darauf hin, dass eine Behauptung von B. Kerékjártó [*Projektív geometria*, Magyar Tud. Akad., Budapest, 1944; MR **9**, 369; S. 371, Zeilen 15–18] irrtümlich ist.

L. Gyarmathi (Debrecen)

2932:

Rollero, Aldo. Alcune osservazioni sulle trasformazioni puntuali fra due S_3. Atti Accad. Ligure **14** (1957), 178–187 (1958). (English summary)

A correspondence between two projective spaces is studied in the neighborhoods of two corresponding points, O, $\bar{O}$ (which are regular for the correspondence). It is well known that there is a collineation ω between the stars of lines with center at O, $\bar{O}$ and (in general) there are seven inflexional lines (in each star). Through each non-inflexional line $r \ni O$ there is one plane π (considered as a second order cap with center at O) to which corresponds a second order cap having $\bar{r}=\omega r$ as asymptotic tangent. Let ω^* be the transformation such that $\pi=\omega^* r$; there are two lines r, r' corresponding to π in ω^*. The correspondence so determined between r and r' is a Geiser involution generated by a homaloidal net of cones of order 8 having as fundamental lines the inflexional lines.

E. Bompiani (Rome)

2933:

Pickert, Günter. Angeordnete projektive Ebenen. Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 41–45. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

In his *Projektive Ebenen* [Springer, Berlin, 1955; MR **17**, 399; p. 227], the author calls an affine plane "ordered"$\Leftrightarrow$ (i) every line has at least three distinct points, and (ii) a betweenness relation [P.E., p. 221] is defined on each range of points—the relation being invariant under every affine projectivity [P.E., p. 12]. (A projectivity of line l to line l' is called "affine"$\Leftrightarrow$ the ideal point of l maps onto the ideal point of l'.) An ordered affine plane can be extended to an "ordered projective plane" so that betweenness leads, in a natural way, to a "separation" relation [P.E., p. 227]. (Conversely, an ordered projective plane can be specialized to an ordered affine plane [P.E., pp. 227–231].)

In the present paper, the author simplifies the definition of "ordered affine plane", replacing "affine projectivity" by "parallel projection". (A "parallel projection" is a product of perspectivities [line onto line], each having an ideal point as center of perspectivity.) The author shows that the new definition is equivalent to the first, so that ordered affine planes can still be extended to ordered projective planes in the usual way. To do this, he starts with an ordered affine plane (relative to parallel projection), establishes the Axiom of Pasch, and defines the two planar "sides" of a line. The final step is to prove the following [P.E., pp. 29, 230]: If $SA_i \| S'A_i'$ $(i=1, 2, 3)$, $SS' \neq A_1A_2 = A_1'A_2' \| SS'$ and A_2 is "between" A_1 and A_3, then A_2' is "between" A_1' and A_3'. This conclusion uses a lemma of S. Crampe: If $SA_i \| S'A_i'$ $(i=1, 2)$, $SS' \neq A_1A_2 = A_1'A_2' \| SS'$ and $A_1 < A_2$, then $A_1' < A_2'$. The relation of $<$ to "betweenness" is described on p. 221 of P.E. The lemma is stated in Crampe's paper, Math. Z. **69** (1958), 435–462 [MR **20** #4806], which was unpublished when the present paper appeared.

The author next considers the Hall ternary, $T(m, x, b)$, and the ordering $<$, on the ternary-system, from which "betweenness" is derived (assuming $0<1$). He asks for a set of properties ("Monotoniegesetzen") which characterize a ternary-system that corresponds to an ordered plane. Crampe [loc. cit.] has described such a set, consisting of two properties: (1) if $b_1<b_2$, then $T(m, x, b_1)<T(m, x, b_2)$; (2) if $T(m_1, x_0, b_1)=T(m_2, x_0, b_2)$ and $m_1<m_2$, then $T(m_1, x, b_1) \lesseqgtr T(m_2, x, b_2)$ according as $x_0 \lesseqgtr x$. Property (1) involves a change in only one argument, but such is not the case for (2). It is reasonable, then, to ask if the characterizing set (1), (2) can be replaced by (1), (2_a), (2_b), where each of (2_a) and (2_b) is a special case of (2) involving a change in only one argument: (2_a) if $m_1<m_2$, then $T(m_1, x, b) \lesseqgtr T(m_2, x, b)$, according as $0 \lesseqgtr x$; (2_b) if $x_1<x_2$, then $T(m, x_1, b) \lesseqgtr T(m, x_2, b)$, according as $0 \lesseqgtr m$.

It is known that (2) may be replaced by (2_a) and (2_b) when the ternary-system forms a quasifield [P.E., p. 237]. Crampe has shown [loc. cit.] that (2_a) and (2_b) may replace (2) if the ternary-system is "conditionally complete". (A system is "conditionally complete" $\Leftrightarrow$ every non-void subset that is bounded above [below] has a greatest lower [least upper] bound.)

W. A. Pierce (Syracuse, N.Y.)

2934:

Bruins, E. M. Metrisation by connexes and curves. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 59–66.

As a generalization of the classical non-euclidean metrical geometry defined by means of a fixed conic, the author defines some other possible metrical geometries by means of connexes and curves as special cases of connexes. With the notation of R. Weitzenböck [*Invariantentheorie*, Noordhoff, Groningen, 1923], let $(au')^m(\alpha'x)^n$, $(bu')^\mu(\beta'x)^n$ be two connexes of the same degree in x; then the distance of the points x and y can be defined by

$$s(x, y) = k \ln \frac{(au')^m(bu')^\mu(\alpha'x)^n(\beta'y)^n}{(au')^m(bu')^\mu(\alpha'y)^n(\beta'x)^n}, \quad u' = (xy).$$

The curves of constant distance to x are of degree $m+\mu+n$, having $(m+\mu)$-tuple points in x. Dually one can use two connexes of the same degree in u' for the angles. Curves of degree >2 provide a great number of comitants which can be used in pairs and give several different metrical geometries. As examples the metrization by two collineations $A=(au')(\alpha'x)$, $B=(bu')(\beta'x)$ and the metrization by the connexes $\theta=(a'b'u')^2(a'x)(b'x)$ and $Q=(a'b'u')^2(a'c'u')(b'x)(c'x)^2$ of a cubic $(a'x)^3=0$, are given in detail. The paper simplifies and completes some results of Martynenko [Ukrain. Mat. Ž. **10** (1958), 251–269; MR **21** #1560]. *L. A. Santaló* (Buenos Aires)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 2645.

2935:

Sperling, Günter. Lösung einer elementargeometrischen Frage von Fejes Tóth. Arch. Math. **11** (1960), 69–71.

Fejes Tóth proposed the problem: given n points on the unit sphere pairwise joined by a shortest arc, for which point distribution is the sum of such arcs a maximum and what is this maximum? Theorem: The sum of all the $k(2k-1)$ mutual spherical distances of $2k$ points of the unit sphere is at most $k^2\pi$. This maximum can be obtained if k of the points coincide at the north pole and k coincide at the south pole. The proof is short, elementary and extends to $2k$ points of the sphere in E_n. *L. M. Kelly* (E. Lansing, Mich.)

2936:

Wesler, Oscar. An infinite packing theorem for spheres. Proc. Amer. Math. Soc. **11** (1960), 324–326.

Theorem: Let $\{C_k\}$ with radii $\{r_k\}$ $(k=1, 2, 3, \cdots)$ be a sequence of disjoint spheres in n-space contained in a unit sphere C_0 and exhausting it but for a set of measure zero, i.e., so that $\sum_{k=1}^{\infty} r_k{}^n=1$. Then $\sum_{k=1}^{\infty} r_k{}^{n-1}=\infty$. The author gives a simple proof using the well-known Borel-Cantelli lemma [see M. Loève, *Probability theory*, van Nostrand, New York, 1955; MR **16**, 598; p. 228]. A proof for the case $n=2$ can be found in a paper by S. N. Mergelyan [Amer. Math. Soc. Transl. No. 101 (1954); MR **15**, 612; p. 21; originally published as Uspehi Mat. Nauk (N.S.) **7** (1952), 31–122; MR **14**, 547]. The author then shows that it is indeed possible to pack a sphere to within a subset of measure zero by means of disjoint subspheres, and indicates that similar theorems hold for more general geometric figures, e.g., a polygon packed by disjoint ellipses, a cube packed by disjoint spheres, etc. *W. Moser* (Winnipeg, Man.)

DIFFERENTIAL GEOMETRY

See also 2801, 2932, 3006, 3007.

2937:

Mihăilescu, Tiberiu. ★Geometrie diferenţială proiectivă. [Projective differential geometry.] Biblioteca Matematică, Vol. II. Editura Academiei Republicii Populare Romîne, Bucharest, 1958. 494 pp. 25.90 lei.

Un traité systématique et détaillé de la géométrie différentielle des courbes, surfaces et variétés non holonomes plongées dans S_3. L'A. utilise les méthodes de Cartan; il suppose la connaissance de la théorie des systèmes en involution; la normalisation du repère est expliquée en détail. La bibliographie (1949–1956) est une continuation incomplète de celle de M. Bol [*Projektive Differentialgeometrie*, I, Vandenhoeck & Ruprecht, Göttingen, 1950; MR **11**, 539]. Table de matières: I. Préliminaires sur les espaces et repères projectifs. II. Courbes planes, courbes de courbure constante. III. Courbes gauches. IV. La théorie générale des surfaces, surfaces réglées et quadriques; particularisation du repère. V. Surfaces non réglées, droites canoniques, quadriques associées, surfaces spéciales. VI. Réseaux conjuguées dans S_3, transformées de Laplace, quadriques et droites associées, réseaux spéciaux. VII. Variétés non holonomes linéaires dans S_3. Eléments associés à une $V_3{}^2$, normales projectives et droites canoniques, quadriques de Lie et Darboux, variétés aux asymptotiques indéterminées ou à invariant fondamental constant ou du type de Tzitzeica-Wilczynski. *A. Švec* (Prague)

2938:

Gallarati, Dionisio. Intorno a certe V_4 di S_8 ed una proprietà caratteristica della $V_4{}^6$ che rappresenta le coppie di punti di due piani. Ricerche Mat. **8** (1959), 52–82.

In questa Memoria si considerano le varietà (differenziabili o) complesse V_4 di dimensione (reale o) complessa 4 regolarmente immerse C' in uno spazio proiettivo (reale o) complesso S_8, che non siano nè coni nè composte di spazi lineari S_4 e i cui S_4 tangenti si appoggiano ad un gruppo di piani a tre a tre indipendenti.—Si considerano dapprima le V_4 cogli S_4 tangenti appoggiati a due piani π_1 e π_2, e quindi le V_4 i cui S_4 tangenti si appoggiano a tre piani π_1, π_2, π_3, supponendo che non esiste un S_5 segato in piani dagli S_4 tangenti della V_4. Si trovano in tal caso due diverse famiglie di V_4 non coni, dipendenti rispettivamente da 4 e 5 funzioni arbitrarie di due parametri. La prima famiglia è costituita dalle serie ∞^2 di piani appoggiati ai tre piani dati.—In questa famiglia l'unica V_4 i cui S_4 tangenti incontrano un quarto piano π_4 assegnato è la $V_4{}^6$ di C. Segre, prodotto cartesiano di due piani. Nella seconda famiglia invece le V_4 i cui S_4 tangenti incontrano anche π_4 sono varietà i cui S_4 tangenti si appoggiano a quattro curve L_i di π_i $(i=1, 2, 3, 4)$ oppure serie ∞^1 di S_3 appoggiati a quattro piani.

Tralasciando vari altri casi esaminati da quanto sopra seguono già i risultati principali della memoria: (I) La $V_4{}^6$ di C. Segre è caratterizzata dall'avere gli S_4 tangenti appoggiati a quattro piani a tre a tre indipendenti colle seguenti ipotesi: (i) non sia conica, (ii) i suoi S_4 tangenti non segano piani su un medesimo S_5, (iii) l'insieme dei punti di appoggio su ciascuno piano è ∞^2. (II) Se si aggiunge un quinto piano π_5 e si considerano le V_4 soddisfacenti le ipotesi (i) e (ii) i cui S_4 tangenti si appoggiano ai cinque piani $\pi_1, \cdots, \pi_5$, si trovano oltre la $V_4{}^6$ di C. Segre anche le serie ∞^1 di S_3 appoggiati ai cinque piani. *F. Gherardelli* (Florence)

2939:

Segre, Beniamino. Sui riferimenti fra superficie per incidenza o parallelismo di piani tangenti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **25** (1958), 381–388.

L'auteur étudie, dans un espace projectif S_n $(n>3)$, les correspondances $\mathcal{T}$ entre deux surfaces (ou deux portions de surfaces) X et Y jouissant de la propriété qu'en un couple quelconque de points homologues x et y les plans tangents à X et Y se coupent suivant une droite r. Se plaçant d'abord dans le cas où l'homographie induite par $\mathcal{T}$ entre les deux faisceaux de tangentes en x et y à X et Y détermine sur r une homographie (π) réduite à l'identité, il caractérise cette dernière circonstance par le fait que les différentes droites (xy) concourent en un point fixe O de S_n, et précise la nature de la correspondance lorsque les droites r sont dans un même hyperplan S_{n-1}, celle-ci étant alors induite par une homologie de centre O et d'hyperplan fondamental S_{n-1}. Dans le cas général l'éventualités suivant laquelle les points unis U et V de (π) sur r sont réels et distincts (cas hyperbolique) l'amène à considérer les réseaux des courbes enveloppées par xU et xV ou yU et yV sur X et Y, courbes dites caractéristiques de $\mathcal{T}$, lesquelles sont les traces, sur X et Y, des deux familles de développables de la congruence des droites (xy), familles qui se confondent dans le cas parabolique, leurs arêtes de rebroussement constituant alors une famille d'asymptotiques de la surface qu'elles engendrent. L'auteur étudie d'une façon approfondie le cas où les droites r appartiennent à un S_{n-1} fixe, pris comme hyperplan de l'infini de S_n considéré comme un espace affine. $\mathcal{T}$ est alors une correspondance par plans tangents parallèles entre X et Y, et la considération du cas $n=3$ conduit à une décomposition en un produit de deux involutions de l'homographie (π), dont la nature fournit une caractérisation, basée sur la considération des asymptotiques de X et Y, des différentes éventualités auxquelles l'étude de $\mathcal{T}$ peut donner lieu. Dans le cas de l'espace général S_n affine l'ensemble des ∞^2 droites (xy) est stratifiable, en ce sens que les ∞^1 surfaces décrites par des points partageant le segment (xy) dans un rapport constant sont parallèles, les surfaces de stratification étant les ∞^1 surfaces en question. Les cas identique, hyperbolique et parabolique sont successivement examinés du point de vue analytique. Le cas hyperbolique conduit entre autre, lorsque X et Y présentent le cas involutif, à une nouvelle présentation de la transformation classique de Moutard, et l'examen du cas parabolique fait dépendre les couples X et Y qui en sont justiciables de deux équations de Laplace du type parabolique, telles que toute solution de l'une détermine une solution de l'autre moyennant de simples quadratures, les couples (X, Y) correspondants admettant une famille d'asymptotiques aux points homologues desquelles les tangentes sont parallèles.

P. Vincensini (Marseille)

2940:

Grincevičius, K. The moving tetrehedron of a complex of straight lines in projective space. Vilniaus Valst. Univ. Mokslo Darbai. Mat. Fiz. Chem. Mokslų Ser. **3** (1955), 25–34. (Lithuanian. Russian summary)

A tetrahedron $A_1A_2A_3A_4$ is canonically associated with a variable ray of a line complex in which at least two of the inflection centers are real. A_1, A_2 are chosen as inflection centers, A_3 $[A_4]$ lies on the tangent plane of the cone of the complex with apex at A_2 $[A_1]$. If $dA_i=\omega_i^kA_k$, $\omega_i^i=0$, then the canonical tetrahedron is determined by 10 linear and 6 bilinear equations in the ω_i^k. From the lines of the complex a ruled surface (*) $\omega_2^3=0$, $\omega_2^4=u\omega_1^3$ is defined, whose tangent plane at A_1 $[A_2]$ is $A_1A_2A_3$ $[A_1A_2A_4]$. Various special cases are considered; for example, if the edge A_1A_3 falls into the asymptotic direction of the ruled surface (*) different from the ruling at A_1, then so does the edge A_2A_4 at A_2. The ruled surface defined by $\omega_1^3=0$, $\omega_2^4=\lambda\omega_2^3$ is also discussed. The general theory is then applied to special types of complexes.

H. Busemann (Los Angeles, Calif.)

2941:

Stephanidis, N. K. Über eine Klasse von Strahlensystemen. Math. Z. **73** (1960), 121–126.

The author defines as M systems all noncylindrical ray systems where the median surface is a minimal surface. He derives a necessary and sufficient condition for an M surface and investigates its geometrical properties.

M. Herzberger (Rochester, N.Y.)

2942:

Бляшке, В. [Blaschke, Wilhelm.] ★Введение в геометрию тканей [Einführung in die Geometrie der Waben]. Translated from the German by M. A. Akivis; edited by I. M. Yaglom. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959. 144 pp. 6 r.

Translation of the 1955 edition [Birkhäuser Verlag, Basel-Stuttgart; MR **17**, 780].

2943:

Hämisch, Paul Werner. Metrische Kontraktionen. Math. Ann. **137** (1959), 136–141.

Dans la présente note, l'auteur indique le bénéfice que l'on peut tirer des méthodes qu'il a développées dans ses travaux précédents [mêmes Ann. **134** (1957), 101–113; **135** (1958), 420–443; MR **20** #1693, 4857] dans le cas où les structures envisagées sont munies d'une métrique.

G. Papy (Brussels)

2944:

Emanuele, Laura. Comportamento di elementi curvilinei e superficiali in trasformazioni di contatto. Boll. Un. Mat. Ital. (3) **14** (1959), 200–205. (English summary)

Consider a contact transformation between two planes. The author restates in analytic form the law of transformation for curvilinear elements of the second order, E_2's, containing a given E_1. He then generalizes (a) by computing the analytic form for the law of transformation of E_3's containing a given E_1 and (b) by taking a contact transformation between two spaces and computing the analytic form for the law of transformation of caps of the second order containing a given element of the first order.

A. Schwartz (New York)

2945:

Speranza, Francesco. Sulla normale affine. Boll. Un. Mat. Ital. (3) **14** (1959), 504–515. (English summary)

The following theorems are proved. (1) If the euclidean and affine normals at a point A of a plane curve coincide, the curve is circular, i.e., $dR^{-1}/ds=0$ (R^{-1}: curvature) at the point A. (2) If the affine and projective normals at a point A of a plane curve coincide, the affine curvature of the curve is a linear function of the affine length σ, i.e., $d^2k/d\sigma^2=0$ (k: affine curvature) at the point A. (3) If t represents the tangent of a surface S which is coplanar with the euclidean and affine normals, the curves enveloped by t form the family which is conjugate to the level curves of

the total curvature of S, i.e., to the curves for which $dK=0$, K being the total curvature. There are indicated also some relations between the affine normal of a surface S and various lines projectively related to S. And the following property is proved: The curves of S enveloped by the tangents which are coplanar with the affine and projective normals constitute the family which is conjugate to the level curves of the affine curvature of S. For non-holonomic hypersurfaces, it is proved that the only ones for which at every point A the directrix line (related affinely to the point A) coincides with the euclidean normal are those which are formed by the normal planes to a line congruence (∞^{r-1} system in an r-dimensional space). Some properties of such non-holonomic hypersurfaces are indicated. *A. Kawaguchi* (Sapporo)

2946:

Sauer, Robert. Geodätisch invariante Kurven bei beliebigen Abbildungen von Flächen. Z. Angew. Math. Mech. **40** (1960), 47–49.

In order to cover second order properties of surface mappings the author studies the class of curves whose geodesic curvature remains unchanged by a given mapping (geodätisch invariante Kurven). These curves are solutions of a nonlinear second order differential equation discussed generally and more in detail for conformal mappings. *F. Stallman* (Washington, D.C.)

2947:

Petrov, P. I. Invariants and classification of differential quadratic forms in four variables. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 387–420. (Russian)

Author's summary: "The author constructs in finite form the simplest basis for a complete system of differential invariants of Riemannian varieties V_4, metrized by a non-singular quadratic differential form $ds^2=g_{ij}(x)dx^idx^j$ with signature $s=4, 0, -2$. Also, by means of arithmetic invariants of the irreducible parts of the curvature tensor R_{hijk} of the fundamental tensor g_{ij} of the space, he gives a classification of the four-dimensional Riemannian spaces themselves. Then by specialization he obtains the invariants and the classification of the four-dimensional conformally-flat Einstein spaces for arbitrary values of the signature s."

2948:

Chae, Younki. Symmetric spaces which are mapped conformally on each other. Kyungpook Math. J. **2** (1959), 65–72.

Let V_n and $\bar{V}_n$ be n-dimensional Riemann spaces whose fundamental tensors are g_{ij} and $\bar{g}_{ij}$ respectively. If a function σ exists such that $\bar{g}_{ij}=e^{2\sigma}g_{ij}$ the spaces are mapped conformally on each other. The author works out the conditions on σ that $\bar{V}_n$ be symmetric, i.e., that the covariant derivative of the Riemann tensor $\bar{R}_{ijkl}$ be identically zero. He then discusses the integrability conditions of the resulting equations. He also deduces the conditions that two symmetric spaces can be mapped conformally on each other.

H. W. Brinkmann (Swarthmore, Pa.)

2949:

Haĭmovič, A. [Haimovici, A.] Equivalence of two spaces with affine connectivity. Dokl. Akad. Nauk SSSR **128** (1959), 1132–1134. (Russian)

L'auteur étudie le problème général de l'équivalence de deux espaces L_n et L_n' localement euclidiens à connexion affine, problème déjà traité dans certains cas particuliers par G. Vrănceanu [Publ. Math. Debrecen **4** (1956), 359–361; MR **18**, 822] et ses élèves. On dit que deux espaces à connexion affine sont équivalents s'il existe entre eux un homéomorphisme deux fois différentiable qui respecte le parallélisme. L'auteur utilise un résultat d'Ostrowski pour transformer le problème et il termine sa note par l'énoncé d'un théorème donnant des conditions suffisantes pour l'équivalence. *M. Decuyper* (Lille)

2950:

Sen, Hrishikes. On relations of curvature tensors over Sen's system of affine connections. Rend. Sem. Mat. Univ. Padova **29** (1959), 242–255.

This paper is concerned with the system of affine connections developed by R. N. Sen [Bull. Calcutta Math. Soc. **42** (1950), 1–13, 177–187; **44** (1952), 92; MR **12**, 205, 533]. For the curvature tensors derived from these connections, the present author proves certain properties which may be considered generalizations of the Bianchi identities and symmetry relations satisfied by a Riemannian curvature tensor. *A. Fialkow* (Brooklyn, N.Y.)

2951:

Dumitraš, V. Determination of the space A_n with group G_r ($r>n^2-n$). Rev. Math. Pures Appl. **4** (1959), 21–42. (Russian)

In accordance with Y. Mutō's classification of projectively flat symmetric A_n [Sci. Rep. Yokohama Nat. Univ. Sect. I **1955**, no. 4, 1–18; MR **17**, 783] certain types of A_n are investigated. (I) Such A_n for which $f=P_{kl}dx^k\delta x^l\equiv 0$, $\varphi=\Gamma_{jl}dx^jdx^l$ of rank two; here $P_{kl}=\Gamma^i_{ikl}$, $\Gamma_{jl}=\Gamma^i_{jil}$; $\Gamma^i_{jkl}=-\Gamma^i_{jlk}$ is the curvature tensor. It is shown that in this case the Γ^i_{jk} can be written $\Gamma^i_{jk}=\delta_j{}^i\varphi_k+\delta_k{}^i\varphi_j$, $\varphi_i=\frac{1}{2}\partial \ln F/\partial x^i$, where F is an arbitrary quadratic form $a_{ij}x^ix^j+a_ix^i+a$, $a_{ij}x^ix^j$ of rank 2. The motion group is the transitive projective group in n variables under which the surfaces $F=0$ are invariant. It has n^2-n+1 parameters. (II) Such A_n for which f is of rank 2 and φ of rank one. In this case also expressions for f, φ and the Γ are derived, as well as for the (n^2-n+1)-parametric motion group. Here certain Pfaffians are invariant.

D. J. Struik (Cambridge, Mass.)

2952:

Dumitraş, V. Maximally mobile spaces with general asymmetrical affine connection. Rev. Math. Pures Appl. **4** (1959), 425–430.

The (generally asymmetric) affine spaces A_n with a maximum motion group, hence with n^2-2n+6 parameters, are the spaces with absolute parallelism and covariant constant torsion tensor. The components of this torsion tensor $t_{bc}{}^a$ can be made zero with the exception of $t_{23}{}^1=-t_{32}{}^1=1$. Such a space is equivalent to the space for which all the coefficients of connection $\Gamma_{bc}{}^a$ are zero with the exception of $\Gamma_{23}{}^1=-\Gamma_{32}{}^1=\frac{1}{2}$.

D. J. Struik (Cambridge, Mass.)

2953:

Dumitraş, V. The classification of the projectively flat spaces A_3. Rev. Math. Pures Appl. 4 (1959), 623–639.

The following cases are systematically investigated [see #2951 for notation]: (a) A_3 with $f\equiv 0$ and φ respectively of rank 1, 2 and 3 (the last ones are Riemannian with constant curvature); (b) A_3 with f of rank 2 and φ of rank 1; (c) A_3 with f of rank 2 and φ of rank 3; and (d) A_3 with both f and φ of rank 2. *D. J. Struik* (Cambridge, Mass.)

2954:

Ōtsuki, Tominosuke. On general connections. I. Math. J. Okayama Univ. 9 (1959/60), 99–164.

In a previous paper [same J. 8 (1958), 143–179; MR **21** #3895], the author developed a theory of general connexions. In the present paper he makes certain changes which increase its range of application.

The operation of covariant differentiation with respect to a general connexion is given by

$$V^{i_1\cdots i_p}_{j_1\cdots j_q,h} = P^{i_1}_{k_1}\cdots P^{i_p}_{k_p}\left(\frac{\partial V^{k_1\cdots k_p}_{h_1\cdots h_q}}{\partial u^h}\right)P^{h_1}_{j_1}\cdots P^{h_q}_{j_q}$$
$$+\sum_{s=1}^{p} P^{i_1}_{k_1}\cdots P^{i_{s-1}}_{k_{s-1}}\Gamma^{i_s}_{k_s h}P^{i_{s+1}}_{k_{s+1}}\cdots P^{i_p}_{k_p}V^{k_1\cdots k_p}_{h_1\cdots h_q}P^{h_1}_{j_1}\cdots P^{h_q}_{j_q}$$
$$-\sum_{t=1}^{q} P^{i_1}_{k_1}\cdots P^{i_p}_{k_p}V^{k_1\cdots k_p}_{h_1\cdots h_q}P^{h_1}_{j_1}\cdots P^{h_{t-1}}_{j_{t-1}}\Lambda^{h_t}_{j_t h}P^{h_{t+1}}_{j_{t+1}}\cdots P^{h_q}_{j_q},$$

where $V^{i_1\cdots i_p}_{j_1\cdots j_q}$ are components of a tensor field of type (p, q), $\Gamma^i{}_{jk}$ are components of a classical affine connexion, $P^i{}_j$ are components of a tensor field of type (1, 1), and $\Lambda^i{}_{jk}=\Gamma^i{}_{jk}-\partial P^i{}_j/\partial u^k$. This generalised operator leads to classical covariant differentiation when $P^i{}_j=\delta^i{}_j$.

It is difficult to summarise adequately the various theorems and lemmas of this lengthy paper. Conditions are found for the commutativity of the operations of generalized covariant differentiation and contraction, and the torsion and curvature forms of a generalized connexion are found. Finally a universal general connexion is introduced, and its curvature and torsion forms are examined. *T. J. Willmore* (Liverpool)

2955:

Michael, J. H. An n-dimensional analogue of Cauchy's integral theorem. J. Austral. Math. Soc. 1 (1959/61), 171–202.

The author considers continuous maps (f, M_n), or f: $M_n\to R^{n+1}$, from a compact n-manifold M_n without boundary into the Euclidean $(x_1, \cdots, x_{n+1})$-space R^{n+1} (closed parametric n-surfaces in R^{n+1}). The set $f(M_n)$ is compact, and by $O(f, M^n)$ is denoted the open bounded subset of R^{n+1} which is the union of all bounded components of $R^{n+1}-f(M_n)$. Real-valued functions $g_1, \cdots, g_{n+1}$ are considered which are continuous in the compact set $f(M_n)\cup O(f, M_n)$, and have the partial derivatives $\partial g_i/\partial x_i$ continuous in $O(f, M_n)$, $i=1, \cdots, n+1$. The usual projection of $(x_1, \cdots, x_{n+1})$ into $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})$ is denoted by P_i, $i=1, \cdots, n+1$. The following assumptions are made: (1) (f, M_n) is of bounded variation [for definition and related restrictions see the paper by the same author in Proc. London Math. Soc. (3) 7 (1957), 616–640; MR **20** #955]; (2) the set $f(M_n)$ has finite n-dimensional Hausdorff measure in R^{n+1}; (3) $\sum(-1)^i(\partial g_i/\partial x_i)=0$ everywhere in $O(f, M_n)$, where $\sum$ ranges over all $i=1, \cdots, n+1$.

Under these hypotheses the Cauchy-type statement is proved that

$$\sum_{i=1}^{n+1}\int_{(f,M_n)} g_i(x)dP_i(x) = 0,$$

where the $n+1$ integrals are the linear n-surface integrals defined by the author in the paper quoted above. *L. Cesari* (Ann Arbor, Mich.)

2956:

Borisov, Yu. F. The parallel translation on a smooth surface. IV. Vestnik Leningrad. Univ. 14 (1959), no. 13, 83–92. (Russian. English summary)

For the first three papers on this subject see same Vestnik **13** (1958), no. 7, 160–171; no. 19, 45–54; **14** (1959), no. 1, 34–50 [MR **21** #3032, 3033, 3034]. In this fourth section a smooth surface $\mathbf{r}=\mathbf{r}(u_1, u_2)$ is considered on which there exists a positive definite metric $g_{ij}du_idu_j$ in a domain D, $g_{ij}\in C^1$. If $\Delta_{u_1}\mathbf{r}_{u_i}=\mathbf{r}_{u_i}(u_1+\Delta u_1, u_2)-\mathbf{r}_{u_i}(u_1, u_2)$ and $\Delta_{u_2}\mathbf{r}_{u_i}=\mathbf{r}_{u_i}(u_1, u_2+\Delta u_2)-\mathbf{r}_{u_i}(u_1, u_2)$, then the two following conditions are equivalent: (A) $\mathbf{r}\in C^1$, $\mathbf{r}_{u_1}\times\mathbf{r}_{u_2}\neq 0$, $(\Delta_{u_j}\mathbf{r}_{u_i}/\Delta u_j)\mathbf{r}_{u_k}\to\left\{\begin{matrix}k\\ ij\end{matrix}\right\}$, uniformly, when $\Delta u_j\to 0$, $(u_1, u_2)\in\varphi\subset D$, Q compact; (B) $\theta^2/\rho_F\to 0$ uniformly, when $\rho_F\to 0$, inside Q. For the surfaces satisfying (B) see paper II. *D. J. Struik* (Cambridge, Mass.)

2957:

Sun', Hè-Sèn. On the uniqueness of the solution of degenerating equations and the rigidity of surfaces. Dokl. Akad. Nauk SSSR 122 (1958), 770–773. (Russian)

In the first part of this paper the author derives rigidity results from results on differential equations; the second part essentially reverses this procedure. Theorem 1 asserts the existence and uniqueness of the solution of a Cauchy problem for the equation

$$y^mf(y)v_{yy}-v_{xx}+a(x, y)v_x+b(x, y)v_y+c(x, y)v = \varphi(x, y),$$

where $0<m<2$. The proof is said to be based on the methods of I. S. Berezin [Mat. Sb. (N.S.) **24 (66)** (1949), 301–320; MR **11**, 112] and Protter [Canad. J. Math. **6** (1954), 542–553; MR **16**, 255]. Three theorems on infinitesimal rigidity of surfaces of revolution are derived from the uniqueness part of this problem. The simplest one to reproduce here is Theorem 3: If from a torus a piece of negative curvature is removed such that no two asymptotic lines starting from a point on the resulting boundary intersect again, then the remaining piece is infinitesimally rigid.

In part 2 the author extends the integral equation method of Grotemeyer [Math. Z. **59** (1953), 258, 278–289; MR **15**, 462] to prove the infinitesimal rigidity of a surface of nonnegative curvature whose simple projection on a plane is bounded by one convex and n concave curves. This result implies, in turn, a uniqueness theorem for the equation

$$r\zeta_{yy}-2s\zeta_{xy}+t\zeta_{xx} = 0$$

under rather complicated boundary conditions. A final result is the uniqueness of the solution, in a rectangle (actually, cylinder) containing a segment of $y=0$ in its interior, of the boundary value problem for

$$y^{2m+1}\zeta_{xx}+\zeta_{yy} = 0,\quad m \geq 0.$$

L. W. Green (Minneapolis, Minn.)

2958:

Belov, K. M. On the uniqueness of surfaces of positive curvature with a boundary. Dokl. Akad. Nauk SSSR **127** (1959), 239–241. (Russian)

Let F_i ($i=1, 2$) be a sufficiently smooth surface in E^3 homeomorphic to a closed disk and with a smooth boundary B_i. Denote by $\tau_i(p)$ the geodesic torsion of B_i at p, by $\nu_i(p)$ the normal curvature of B_i at p, and by $H_i(p)$ the mean curvature of F_i at p. Finally assume that F_i has positive Gauss curvature (including B_i) and that an intrinsically isometric mapping f of F_1 on F_2 exists. If either $\tau_1(p)=\tau_2(f(p))$, or $\nu_1(p)-H_1(p)=\nu_2(f(p))-H_2(f(p))$ for all $p \in B_1$, then F_1 and F_2 are congruent, i.e., f can be extended to an isometry of E^3 on itself. It follows, for example, that F_1 and F_2 are congruent if B_1 and B_2 are lines of curvature or consist only of umbilical points.

H. Busemann (Los Angeles, Calif.)

2959:

Pogorelow, A. W. Die eindeutige Bestimmung allgemeiner konvexer Flächen. Deutsch. Akad. Wiss. Berlin. Schr. Forschungsinst. Math. **3** (1956), 79 pp.

A translation from the Russian [Monografii Instituta Matematiki, 2. Akad. Nauk Ukrainskoĭ SSR, Kiev, 1952; MR **16**, 162].

2960:

Погорелов, А. В. [Pogorelov, A. V.] ★Бесконечно малые изгибания общих выпуклых поверхностей. [Infinitesimal bending of general convex surfaces]. Izdat. Har'kov. Gos. Univ., Kharkov, 1959. 106 pp. 4.15 r.

This booklet contains the complete proofs for the results stated in Dokl. Akad. Nauk SSSR **128** (1959), 475–477 [MR **21** #7542]. These follow comparatively quickly from the following lemma, whose proof occupies a large part of the book. Let $z=z(x, y)$ represent a convex surface not containing plane pieces. Let $\zeta(x, y)$ be the z-component of a vector field on the surface defining an infinitesimal isometry. Then the surface $S: z=\zeta(x, y)$ has negative curvature in the sense that no point p of S lies in a plane H such that a neighborhood of p in H contains no other points of S than p.

H. Busemann (Los Angeles, Calif.)

2961:

Boyarskiĭ, B. V. On the rigidity of some composite surfaces. Uspehi Mat. Nauk **14** (1959), no. 6 (90), 141–146. (Russian)

This article examines the rigidity of closed surfaces consisting of convex pieces. It is proved (Theorem I) that: if S_1 and S_2 are convex surfaces of revolution, homeomorphic to a disk, with a common parallel γ, situated on one side of the plane of this parallel and without common points other than the points of γ, then the closed surface Σ formed from S_1 and S_2 by sewing along the curve γ is rigid. Each of the surfaces S_i is assumed to be piecewise twice differentiable, and the problem of rigidity is solved with respect to continuous twice differentiable bending fields on each regular piece of the surfaces S_i. Theorem 2 deals with the rigidity of a surface of revolution consisting of three convex surfaces. Theorem 3 establishes the rigidity of a closed surface Σ consisting of an arbitrary convex surface S with n holes patched by convex surfaces S_i. It is required that the surface Σ be a one-valued projection from a certain point and that the surface S_i be convex toward this point. By means of the integral formula of W. Blaschke the rigidity of the surface Σ is established under certain conditions on the seam lines of S and S_i required in order to apply Blaschke's formula to the proof of the theorem.

A. V. Pogorelov (Kharkov)

2962:

Boyarskiĭ, B. V.; Efimov, N. V. The maximum principle for infinitesimal bendings of piecewise-regular convex surfaces. Uspehi Mat. Nauk **14** (1959), no. 6 (90), 147–153. (Russian)

As is well known, for a regular (twice differentiable) surface $z=z(x, y)$, no part of which is planar, the component ζ (along the z-axis) of an infinitesimal bending field $\tau(\xi, \eta, \zeta)$ attains its maximum and its minimum on the boundary of the surface. The authors extend this result to the case of piecewise twice differentiable convex surfaces and bending fields. The proof is based on an investigation of the infinitesimal bending field at the points where the surface fails to be twice differentiable and is carried out with the aid of the equation of infinitesimal bending $dxd\xi+dyd\eta+dzd\zeta=0$. It is pointed out that with the aid of the result so obtained it is possible to prove the rigidity of closed convex piecewise twice differentiable surfaces by application of the method explained in the article of the reviewer [A. V. Pogorelov, Trudy Mat. Inst. Steklov. **29** (1949); MR **12**, 437].

A. V. Pogorelov (Kharkov)

2963:

Bublik, B. A. An example of a system of non-rigid smooth closed surfaces with two linearly independent infinitesimal bendings. Uspehi Mat. Nauk **14** (1959), no. 6 (90), 155–158. (Russian)

S. E. Cohn-Vossen has constructed an example of a closed surface admitting at least one non-trivial infinitesimal bending field and has asked the question whether there exists a surface with several such fields. In solution of this problem the present author constructs an example of a surface of revolution admitting two independent bending fields. A meridian of this surface is represented by a smooth piecewise twice differentiable curve given by the equations

$$\begin{aligned} r(u) &= (2\rho u-u^2)^{1/2} && (0 \leq u \leq a),\\ r(u) &= A \cosh 3^{-1/2}(u-a-c) && (a < u \leq a+c),\\ r(u) &= r(2a+2c-u) && (a+c \leq u \leq 2a+2c), \end{aligned}$$

where ρ, a, c, A are suitably chosen constants. In the construction of this example use is made of the expansion in trigonometrical series of a point-vector of the surface and of the bending field.

A. V. Pogorelov (Kharkov)

2964:

Poznyak, È. G. The relation between non-rigidity of first and second order for surfaces of revolution. Uspehi Mat. Nauk **14** (1959), no. 6 (90), 179–184. (Russian)

The following theorem is proved: If a closed surface of revolution has a finite number of independent infinitesimal bendings of first order, then this surface admits an infinitesimal bending of second order. The proof is based upon the following consideration. Let u, v be a geographic

parametrization of the surface (u = const are the parallels of latitude, v = const are the meridians of longitude); then the system of independent fields of infinitesimal bendings of the surface can be represented in the form $z_k = \exp(is_k)\bar{r}_k(u)$ (s_k integers). It is proved that if the number m occurs among the integers s_k but not the number $2m$ (such an m can always be found if the number of independent infinitesimal bendings of the first order is finite), then the infinitesimal bending of the first order z_m may be continued into an infinitesimal bending of the second order. *A. V. Pogorelov* (Kharkov)

2965:

Kaul, R. N. Curvatures in Finsler space. Bull. Calcutta Math. Soc. **50** (1958), 189–192.

Consider a curve C in an n-dimensional Finsler space. Let $b_{(j)}{}^i$ ($j = 1, 2, \cdots, n$) represent the unit tangent and $(n-1)$ unit normals of C at a point P and let $b_{(j)}{}^{*i}$ be the corresponding vectors at a nearby point P^*. Denote by γ_k ($k = 1, 2, \cdots, n-1$) the curvature of C at P, by δs the arc length PP^*, and by $\delta\theta_j$ the angle at P^* between $b_{(j)}{}^{*i}$ and the vector obtained by parallel displacement of $b_{(j)}{}^i$ from P to P^* along C. The author proves that $\lim_{\delta s \to 0} (\delta\theta_j/\delta s)^2 = \gamma_j^2 + \gamma_{j-1}^2$ ($\gamma_0 = \gamma_n = 0$; $j = 1, 2, \cdots, n$). This result is a generalization of a similar theorem for a curve in a Riemann space proved using analogous methods by H. Levy [Bull. Amer. Math. Soc. **40** (1934), 75–78]. *A. Fialkow* (Brooklyn, N.Y.)

2966:

Barthel, Woldemar. Natürliche Gleichungen einer Kurve in der metrischen Differentialgeometrie. Arch. Math. **10** (1959), 392–400.

In a three-dimensional C^4-manifold M let a length be defined by a function $F(x, X)$ satisfying the usual conditions for a Finsler metric ($F(x, X) > 0$ for $X \neq 0$, $F(x, kX) = kF(x, X)$ for $k \geq 0$, $\sum g_{ik}(x, X)X^iX^k$ positive definite with $g_{ik} = \partial^2 \frac{1}{2}F^2/\partial X^i \partial X^k$). Also let an area be defined by a function $f(x, B)$ defined on the tangential bivectors B of M and satisfying analogous conditions to those for F. Finally let a volume be defined.

It is further assumed that an invariant differential of a vector field $DX^i = dX^i + X^k\gamma_{kh}{}^i(x, X)dx^h$ is given for which parallel displacement preserves length. (Such differentials exist.) An analogous assumption is made regarding the existence of an invariant differential DB^i, and a parallel displacement ($DB^i = 0$) of bivectors leaving area invariant.

Then a theory of curves can be developed. If $T^i(s) = dx^i/ds$ is the unit tangent vector of curve $x(s)$, the curvature $\kappa(s)$ is defined by $DT^i/ds = \kappa(s)H^i(s)$, $\kappa(s) \geq 0$, $f(x, T \wedge H) = 1$. The torsion $\tau(s)$ is defined by a similar modification of the classical definition. The principal result states that, given $\kappa(s) > 0$, $\tau(s) \neq 0$ in a neighborhood of $s = 0$ and initial values for x, T, H, then exactly one curve $x(s)$ exists with s as arc length, $\kappa(s)$ as curvature, $\tau(s)$ as torsion and with the given initial values.

H. Busemann (Los Angeles, Calif.)

2967:

Soós, Gy. Über die homothetische Gruppe von Finslerschen Räumen. Acta Math. Acad. Sci. Hungar. **10** (1959), 391–394. (Russian summary, unbound insert)

The following theorem is proved by making use of Lie derivatives: If a Finsler space F_n has an $(r+1)$-parameter group H_{r+1} of homothetic transformations, then H_{r+1} contains an r-parameter group G_r of motions which is a normal subgroup of H_{r+1}. As a remark the author states also the corollaries: (1) If an F_n admits a homothetic group H_k with $k > \frac{1}{2}n(n-1) + 2$, F_n is a Riemannian space of constant curvature. (2) In an F_n there exists no proper homothetic group H_k with $\frac{1}{2}n(n+1) > k > \frac{1}{2}n(n-1) + 2$.

A. Kawaguchi (Sapporo)

2968:

Varga, O. Über die Zerlegbarkeit von Finslerschen Räumen. Acta Math. Acad. Sci. Hungar. **11** (1960), 197–203. (Russian summary, unbound insert)

In the notation of the author Latin indices run from 1 to n, Greek indices with one and two primes from 1 to r and from $r+1$ to n respectively ($r < n$). The following definition is introduced. An n-dimensional Finsler space F_n with metric function $L(x^j, v^j)$ is said to be a product space of an F_r and an F_{n-r} if there exist two metric functions $L_1(x^{\rho'}, v^{\rho'})$, $L_2(x^{\sigma''}, v^{\sigma''})$ such that $L^2 = (L_1)^2 + (L_2)^2$, where $L_1(x^{\rho'}, 0) = 0$, $L_2(x^{\sigma''}, 0) = 0$. In such a product space F_n two tensors p_{ik}' and p_{ik}'' may be defined by putting $p'_{\rho'\sigma'} = g_{\rho'\sigma'}$, $p'_{\rho''\sigma''} = 0$; $p''_{\rho'\sigma'} = 0$, $p''_{\rho''\sigma''} = g_{\rho''\sigma''}$, where the g's are the metric tensors of F_r and F_{n-r}. The tensors p_{ik}', p_{ik}'' define two mutually orthogonal vector spaces V_r and V_{n-r}, while $Dp_{ik}' = Dp_{ik}'' = 0$. If for two vector fields $\xi^{\rho'}$, $\eta^{\rho'}$ of F_r the field resulting from the construction of the commutator $(\partial\xi^{\rho'}/\partial x^{\sigma'})\eta^{\sigma'} - (\partial\eta^{\rho'}/\partial x^{\sigma'})\xi^{\sigma'}$ is also a field of V_r, the commutator operation is said to be closed with respect to V_r (and similarly for V_{n-r}). It is shown that (a) the existence of a positive semi-definite symmetric tensor p_{ik}' of rank r with $Dp_{ik}' = 0$, and (b) the property of the commutator operation being closed with respect to the V_r defined by p_{ik}' (as well as with respect to V_{n-r}), are necessary conditions for the representability of F_n as a product space. The major part of the paper is devoted to the proof of the theorem according to which these conditions are also sufficient. On the other hand, the existence of vector spaces V_r which may be subjected to unlimited parallel displacements is not sufficient to ensure such decomposability of Finsler spaces.

H. Rund (Durban)

2969:

Hu, Hou-sung. A Finslerian product of two Riemannian spaces. Sci. Record (N.S.) **3** (1959), 446–448.

The author calls a Finsler space F_n a product space of two Riemannian spaces R_m and R_{n-m} if the element of distance in F_n is representable in the form $ds^2 = 2F(p_1, p_2)$, where p_1, p_2 represent the fundamental quadratic forms of R_m and R_{n-m} respectively and where F is homogeneous of the first degree in p_1 and p_2. It is stated that such a Finsler space admits two families of mutually orthogonal geodesic subspaces V_m and V_{n-m}. *H. Rund* (Durban)

2970:

Ermakov, Yu. I. Spaces X_n with an algebraic metric and semimetric. Dokl. Akad. Nauk SSSR **128** (1959), 460–463. (Russian)

Let $F_n{}^{(q)}$ be an n-dimensional Finsler space with the metric given by the differential form of order q: $ds^q =$

$a_{\alpha_1\cdots\alpha_q}d\xi^{\alpha_1}\cdots d\xi^{\alpha_q}$ $(\alpha, \beta, \cdots = 1, 2, \cdots, n)$. The resultant of the n forms $q^{-1}\partial(a_{\alpha_1\cdots\alpha_q}x^{\alpha_1}\cdots x^{\alpha_q})/\partial x^\lambda$ is named the discriminant $\mathfrak{A}$, which is a homogeneous polynomial of dimension $n(q-1)^{n-1}$ in the coefficients $a_{\alpha_1\cdots\alpha_q}$ and a scalar density of weight $w=q(q-1)^{n-1}$. In the present paper $\mathfrak{A}$ is assumed to be different from zero. The differential geometry in $F_n^{(q)}$, especially the determination of the affine connection $\Gamma_{\beta\gamma}{}^\alpha(x)$ under the condition that the covariant derivatives of the fundamental tensor $a_{\alpha_1\cdots\alpha_q}$ be equal to zero, was studied for $F_2^{(q)}$ by A. E. Liber [Trudy Sem. Vektor. Tenzor. Anal. **9** (1952), 319–350; MR **14**, 688], for $F_3^{(3)}$ by K. Tonooka [Tensor (N.S.) **6** (1956), 60–68; MR **18**, 232] and Yu. I. Ermakov [Dokl. Akad. Nauk SSSR **118** (1958), 1070–1073; MR **20** #3578]. C. M. Cramlet [Ann. of Math. (2) **31** (1930), 134–150] also studied the invariants of $F_n^{(q)}$. The present paper is devoted to determining the symmetric affine connection in $F_n^{(q)}$ for $n \geqq 2$, $q > 3$, from which the complete system of differential and conformal differential comitants can be found. In fact, we have the relation $a^{\beta\alpha_2\cdots\alpha_q}a_{\gamma\alpha_2\cdots\alpha_q} = \delta^\beta_\gamma$, where $a^{\alpha_1\cdots\alpha_q} = (q/w\mathfrak{A})\partial\mathfrak{A}/\partial a_{\alpha_1\cdots\alpha_q}$. Put $B^{\sigma\tau\alpha}_{\beta\gamma\omega} = q\delta^\sigma_{(\beta}\delta^\tau_\gamma a_{\rho_3\cdots\rho_q)\omega}a^{\alpha\rho_3\cdots\rho_q}$, which is symmetric in σ, τ and β, γ; then under the assumption Det $(B^{\sigma\tau\alpha}_{\beta\gamma\omega}) \neq 0$ (formed with respect to indices $(\sigma \leqq \tau;\ \omega)$ and $(\beta \leqq \gamma;\ \alpha)$), we can find $P^{\mu\nu\omega}_{\sigma\tau\lambda}$ defined by the equations $B^{\sigma\tau\alpha}_{\beta\gamma\omega}P^{\mu\nu\omega}_{\sigma\tau\lambda} = \delta^{(\mu}_\beta\delta^{\nu)}_\gamma\delta^\alpha_\lambda$, $B^{\sigma\tau\alpha}_{\beta\gamma\omega}P^{\beta\gamma\lambda}_{\mu\nu\alpha} = \delta^{(\sigma}_\mu\delta^{\tau)}_\nu\delta^\lambda_\omega$. The symmetric affine connection can be determined from the condition $\nabla_{(\beta}a_{\gamma\rho_3\cdots\rho_q)}a^{\alpha\rho_3\cdots\rho_q} = 0$ and has the form $\Gamma^\lambda_{\mu\nu} = P^{\beta\gamma\lambda}_{\mu\nu\alpha}(\partial_\beta a_{\gamma\rho_3\cdots\rho_q})a^{\alpha\rho_3\cdots\rho_q}$. The curvature tensor is calculated too. The following theorems are then stated. (1) For $F_n^{(q)}$ to be flat, it is necessary and sufficient that the covariant derivatives of the fundamental tensor $a_{\alpha_1\cdots\alpha_q}$ be zero. (2) Any differential comitant of order p of the tensor $a_{\alpha_1\cdots\alpha_q}$ can be represented by an algebraic comitant of $a_{\alpha_1\cdots\alpha_q}$ and its covariant derivatives of order up to p. The conformal connection $\overset{(k)}{\Gamma}{}^\lambda_{\mu\nu}$ is also determined which is invariant under the conformal change ${}^*a_{\alpha_1\cdots\alpha_q} = \sigma a_{\alpha_1\cdots\alpha_q}$, and similar theorems are stated. Finally, the theory is extended to the case that the pseudo-tensor $A_{\alpha_1\cdots\alpha_q}$ takes the place of the tensor $a_{\alpha_1\cdots\alpha_q}$. *A. Kawaguchi* (Sapporo)

GENERAL TOPOLOGY, POINT SET THEORY

See also 2516.

2971:

Majumdar, N. C. Bose. On the distance set of the Cantor "Middle Third" set. Bull. Calcutta Math. Soc. **51** (1959), 93–102.

Let C be the Cantor ternary set on $[0, 1]$. It is a well-known theorem of Steinhaus [see, e.g., A. Zygmund, *Trigonometric series*, Vol. 1, Cambridge Univ. Press, New York, 1959; MR **21** #6498; p. 235] that for every number d between 0 and 1 there are two points x and y of C that are d apart. The author asks how many such pairs there are, and shows that for all but a countable number of d in C, and for a dense set of d not in C, the points $x \in C$ for which $y \in C$ and $d = x - y$ has the cardinal number of the continuum. {Considerably shorter proofs can be given.} *R. P. Boas, Jr.* (Evanston, Ill.)

2972:

Jones, F. B. On the first countability axiom for locally compact Hausdorff spaces. Colloq. Math. **7** (1959), 33–34.

The author shows that if U is an open subset of a compact Hausdorff space having $\aleph_1$ elements, then the first countability axiom holds at some point of U. As an immediate consequence of this and the Birkhoff-Kakutani theorem on metrizability of groups he obtains the result of A. Hulanicki [Fund. Math. **44** (1957), 156–158; MR **19**, 1063] that a locally compact topological group having $\aleph_1$ elements is metric. *K. deLeeuw* (Princeton, N.J.)

2973:

Šedivá, Věra. On collectionwise normal and hypocompact space. Czechoslovak Math. J. **9 (84)** (1959), 50–62. (Russian. English summary)

In the first part, the following theorems are proved: (1) a normal space is collectionwise normal and countably paracompact [i.e., "strongly normal" in the reviewer's terminology, Colloq. Math. **6** (1958), 145–151; MR **21** #2219] if and only if, for any locally finite $\{B_\lambda\}$, there are open $G_\lambda \supset B_\lambda$ such that $\{G_\lambda\}$ is locally finite; (2) strong normality is preserved under multiplication by a compact metrizable space; (3) collectionwise normality is hereditary for F_σ-subsets. In the second part, hypocompact (every open covering is refined by a star-finite one) spaces are considered. Some of the results may be stated as follows. (a) If X is hypocompact, $Y \subset X$ is an F_σ-set, and every $y \in Y$ has a neighborhood (in Y) of the form $Y \cap H$, H connected, then Y is hypocompact. (b) Let $X = X_1 \times X_2$ be metrizable; if X_1, X_2 are locally separable, then X is hereditarily hypocompact; if X_1 is not locally separable, then (i) hypocompactness of X implies that every closed connected subset of X_2 is compact, (ii) hereditary hypocompactness of X implies ind $X_2 = 0$, where ind denotes the "small" inductive dimension. This last result disproves an assertion by K. Iséki [Portugal. Math. **13** (1954), 149–152; MR **16**, 1138]. *M. Katětov* (Prague)

2974:

Frolík, Zdeněk. On G_δ-spaces. Czechoslovak Math. J. **9 (84)** (1959), 63–65. (Russian summary)

Let P be a topological space. An open base $\mathfrak{B}$ of P is called a V-base if there exist open bases $\mathfrak{B}_n \subset \mathfrak{B}$, $n = 1, 2, \cdots$, such that (i) $\mathfrak{B}_n \supset \mathfrak{B}_{n+1}$, (ii) if a family $\{G\}$ of open sets has the finite intersection property and contains sets from every $\mathfrak{B}_n$, then $\bigcap \bar{G} \neq \emptyset$. A space R is called an extension of P if $R \supset P$, $\bar{P} = R$. Some theorems are stated (without proof) including the following results (for a completely regular P): P is a G_δ-space [i.e., a G_δ-set in every Hausdorff extension; cf. E. Čech, Ann. of Math. (2) **38** (1937), 823–844] if and only if it has a V-base; P is a Baire set in every completely regular extension if and only if $P = S \cup T$ where S is a G_δ-space, T is a meager subset. *M. Katětov* (Prague)

2975:

Ivanova, V. M. Spaces of closed subsets of compact extensions. Mat. Sb. (N.S.) **50 (92)** (1960), 91–100. (Russian)

The author is concerned with obtaining the space of closed subsets (in the Michael topology) of a Shanin

compactification of a space E as a Shanin compactification of the space of closed subsets of E. For the case of the Wallman compactification, she does so.

J. Isbell (Seattle, Wash.)

2976:

Landsberg, Max. Moore-Smith-Folgen und vollständige uniforme Räume. Wiss. Z. Tech. Hochsch. Dresden **8** (1958/59), 675–677.

The author attempts to replace Condry systems by Condry-Moore-Smith sets for the definition of completeness of uniform spaces. His conclusions appear to the reviewer to be easy deductions from well-known facts.

H. Nakano (Kingston, Ont.)

2977:

Ginsburg, Seymour; Isbell, J. R. Some operators on uniform spaces. Trans. Amer. Math. Soc. **93** (1959), 145–168.

Consider the class of all uniform spaces μX, where X is a topological space and μ is a uniformity defined as a family of coverings satisfying well-known conditions. On this class, functions π, f, e, n, s, c, λ are defined [the first three of them are well known; for c, cf. J. Isbell, Pacific J. Math. 8 (1958), 67–86, 941; MR **20** #4261]: $\pi\mu X$ is the completion of μX; $f\mu$ [respectively, $e\mu$, $s\mu$, $n\mu$] has a basis consisting of all finite [countable, countable star-finite, of non-measurable power] coverings $u \in \mu$; $c\mu$ has a basis consisting of all $\varphi^{-1}(v)$ where φ is a uniformly continuous mapping of μX into an Euclidean space R^n, and v is a uniform covering of R^n. The definition of λ is more complicated: if ν is a family of coverings, then $\nu^{(1)}$, the "derivative" of ν, is the family of all coverings refined by some $\{U_\alpha \cap W_\beta{}^\alpha|$ all α, all $\beta\}$ where $\{U_\alpha\}$ and, for each fixed α, $\{W_\beta{}^\alpha\}$ are in ν [for a related operation cf. J. Isbell, loc. cit.]. Families $\nu^{(\alpha)}$, α an ordinal, are defined by induction; the union of all $\nu^{(\alpha)}$ is denoted $\lambda\nu$, and it is proved that $\lambda\nu$ is a uniformity whenever ν is so. The uniformity $\lambda\mu$ may be defined also as the coarsest locally fine uniformity finer than μ (a uniformity is called locally fine if every uniformly locally uniform covering is uniform; μ is locally fine if and only if $\lambda\mu = \mu$).

Various theorems concerning the above functors and related questions are proved, e.g., (1) $c\pi\lambda n = \pi c\lambda n$ (especially, if a locally fine μX is of a non-measurable power, then its completion is determined by uniformly continuous real-valued functions [cf. T. Shirota, Osaka Math. J. 4 (1952), 23–40; MR **14**, 395]); (2) if μX is locally fine, then every uniformly continuous real-valued function on a subspace has an extension uniformly continuous on μX. There are a number of other interesting results of various kinds, including the description of relations (called "uniform") generating uniform quotient spaces (of a given μX).

M. Katětov (Prague)

2978:

Leader, S. On clusters in proximity spaces. Fund. Math. **47** (1959), 205–213.

The author calls a "cluster", in a proximity space, every collection $\mathfrak{c}$ of subsets such that (a) if $A \in \mathfrak{c}$, $B \in \mathfrak{c}$, then A is close to B; (b) if A is close to C for every $C \in \mathfrak{c}$, then $A \in \mathfrak{c}$; (c) if $A \cup B \in \mathfrak{c}$, then either $A \in \mathfrak{c}$ or $B \in \mathfrak{c}$. By means of clusters, several known theorems (concerning compactifications of proximity spaces) are proved; an (apparently new) version of the Stone-Weierstrass theorem is given for proximity spaces.

M. Katětov (Prague)

2979:

Mrówka, S. On the sets of quasi-components. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 703–705. (Russian summary, unbound insert)

The author shows that the class of all quasi-components of a set A, lying in a locally connected separable metric space, can be represented as the inverse limit of the classes of components of all open super-sets of A. This answers a question attributed to K. Kuratowski.

J. Segal (Seattle, Wash.)

2980:

Borsuk, K. On a metrization of polytopes. Fund. Math. **47** (1959), 325–341.

A space X endowed with a metric ρ is said to be strongly convex provided that each pair of points $x, y \in X$ has exactly one midpoint z ($\rho(x, y) = 2\rho(x, z) = 2\rho(z, y)$). (X, ρ) is locally strongly convex (l.s.c.) provided that each point $x \in X$ has arbitrarily small neighborhoods U which are strongly convex in the metric ρ. The main problem concerning l.s.c. is to know whether each compact polyhedron P admits a metric ρ such that (P, ρ) is l.s.c. [K. Borsuk, Jber. Deutsch. Math. Verein. **60** (1958), Abt. 1, 101–114; MR **19**, 1186; p. 108].

In this paper each polyhedron P of a finite dimension n is given a specific metric ρ, called "spherical metric" and defined as follows. Each k-dimensional simplex Δ from a fixed triangulation T of P is mapped affinely onto a k-dimensional face of a regular $(n+1)$-simplex inscribed in the unit sphere S^n of E^{n+1} and then projected into S^n from the center of the sphere. The spherical metric of S^n induces then a "spherical metric" on Δ. The "spherical distance" of two arbitrary points x and y of P (P connected) is obtained by considering all sequences $x = x_0$, x_1, $\cdots$, $x_{m+1} = y$ of points, any two consecutive points x_i, x_{i+1} belonging to the same simplex Δ_i of T. To each such sequence is assigned the sum of the spherical distances from x_i to x_{i+1} (with respect to Δ_i), and $\rho(x, y)$ is the inf of these sums. It appears that this metric is a good candidate to make the polyhedron P l.s.c. Main theorem: Every compact polyhedron P of dimension $\dim P \leq 2$ metrized by the described spherical metric ρ is l.s.c.

S. Mardešić (Zagreb)

2981:

Sieklucki, K. On superpositions of simple mappings. Fund. Math. **48** (1959/60), 217–228.

A mapping f of a space X onto a space Y is of order $\leq k$ if for every point y in Y the set $f^{-1}(y)$ contains at most k points. Mappings of order ≤ 2 are termed simple. By considering relationships between a continuous mapping and the upper semi-continuous decomposition of its domain of definition which it generates the author proves that every continuous mapping of order $\leq k$ defined on a compact n-dimensional space X is a finite superposition of simple mappings; his proof yields an upper bound in terms of k and n for the number of simple mappings required. He utilizes properties of paratactic rotations of the sphere to define a continuous mapping of finite order

on a compact infinite-dimensional space which cannot be written as a finite superposition of simple mappings.

P. V. Reichelderfer (Columbus, Ohio)

2982:

Ponomarev, V. I. On closed mappings. Uspehi Mat. Nauk **14** (1959), no. 4 (88), 203–206. (Russian)

Let a continuous function $f:X \to Y$ be called a B-mapping if it maps closed sets onto closed sets, and if $f^{-1}(y)$ is bicompact for each y in Y. Let $f: X \to Y$ be a B-mapping, and suppose X is normal. Then X is a G_δ in each bicompactification whenever Y is a G_δ in some bicompactification; and X is compact whenever Y is compact. This latter result can be stated as follows. Let D be an upper-semi-continuous decomposition of a normal space X into compact sets. Then X is compact if D is compact.

R. Arens (Los Angeles, Calif.)

2983:

Harrold, O. G., Jr. Locally peripherally euclidean spaces are locally euclidean. Bull. Amer. Math. Soc. **66** (1960), 194–197.

Conditions on a family S of subsets of a Peano space X are given such that if each point of X has a local basis consisting of open sets whose boundaries are in S, then X is a topological 3-sphere. It is stated that the family of tame 2-spheres in the topological 3-sphere is such a family. The conditions on S are then "localized" to give conditions on families S' such that if each point of X has a local basis consisting of open sets whose boundaries are in S', then X is a topological 3-manifold.

R. Ellis (Philadelphia, Pa.)

2984:

Kneser, Hellmuth. Sur les variétés connexes de dimension 1. Bull. Soc. Math. Belg. **10** (1958), 19–25.

A complete classification is given of all possible 1-dimensional manifolds, including the non-separable ones. It is shown that every connected 1-dimensional manifold is either a topological circle, real number line, a half-line or a line of Alexandroff.

G. T. Whyburn (Charlottesville, Va.)

2985:

Goblirsch, R. P. On decompositions of 3-space by linkages. Proc. Amer. Math. Soc. **10** (1959), 728–730.

A finite set of disjoint simple closed curves in euclidean 3-space E^3 is called a linkage, and the decomposition space formed from E^3 by identifying the points on each curve of a linkage is called the space of the linkage. R. H. Bing and M. L. Curtis [Amer. Math. Soc. Notices **5** (1958), 685] showed a linkage of nine circles whose space is not imbeddable in E^4, and conjectured that a linkage of three circles, each pair of which is linked, would have a space that is not imbeddable in E^4. Examples are given to show that this conjecture is incorrect.

R. L. Wilder (Ann Arbor, Mich.)

2986:

Sanderson, D. E. Isotopy in 3-manifolds. III. Connectivity of spaces of homeomorphisms. Proc. Amer. Math. Soc. **11** (1960), 171–176.

[For part II see Duke Math. J. **26** (1959), 387–396; MR **21** #5956.] The following results are proved with a compact-open topology on the spaces of homeomorphisms involved. (1) Given a positive number ε, every homeomorphism h of E^3 onto itself is ε-isotopic to a piecewise linear homeomorphism. (The diameter of the path of each point under an ε-isotopy is less than ε.) (2) The space $H(E)$ of homeomorphisms of E^3 onto itself is locally arcwise connected. (3) $H(E)$ consists of exactly two components. (4) The space $H(S)$ of self-homeomorphisms of the 3-sphere, S^3, is uniformly locally arcwise connected.

S. S. Cairns (Urbana, Ill.)

ALGEBRAIC TOPOLOGY

See also 2516, 2642, 2980.

2987a:

Grötzsch, Herbert. Zur Theorie der diskreten Gebilde. I. Elementare kombinatorische Eigenschaften gewisser Dreikantnetze auf der Kugel und der einfach punktierten Kugel. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **5** (1955/56), 839–844.

2987b:

Grötzsch, Herbert. Zur Theorie der diskreten Gebilde. II. Ein Satz über Vierkantnetze auf der Kugel, mit Anwendung auf beliebige Netze und halbgerade Dreikantnetze auf der Kugel. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **6** (1956/57), 697–704.

2987c:

Grötzsch, Herbert. Zur Theorie der diskreten Gebilde. III. Kongruenzklassen von Dreikantnetzen auf der Kugel und diesbezügliche Fragestellungen. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **6** (1956/57), 785–788.

2987d:

Grötzsch, Herbert. Zur Theorie der diskreten Gebilde. IV. Beweis des Eckentransformationssatzes (2, 1) für Dreikantnetze auf der Kugel. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **6** (1956/57), 789–798.

This review covers the first four of a series of 9 papers all concerned with various combinatorial properties of certain classes of graphs on a sphere or on a plane. The concern throughout is with finite graphs on a sphere which "cover" the sphere or with possibly infinite graphs covering the plane whose vertices have no accumulation element. The author observes that many of the results have been long known to him but do not seem to be in the literature. The relevance of many of the results to various formulations of the four-color problem will be apparent to those familiar with the techniques employed in its analysis.

I. A graph all of whose vertices are of order three is a 3-net (Dreikantnetze). If θ is the greatest common divisor of the n_i, where n_i is the number of vertices in region G_i, then the 3-net is designated by the symbol $N_{3,\theta}$. Theorem: Each $N_{3,\theta}$, $\theta > 1$, is regular, i.e., each edge belongs to two different regions. Theorem: If $\theta \neq 3$, then the numbers $1, 2, \cdots, \theta$ can be assigned to the vertices of $N_{3,\theta}$ in such a way that in each region θ consecutive vertices carry

equivalent permutations of the number $1, 2, \cdots, \theta$; furthermore, starting at any point the permutations occur periodically. A somewhat similar theorem holds for what the author calls the edge-signature function.

II. The study of vertex signature functions continues, leading to the following. Theorem: In an arbitrary 4-net on the sphere with alternately signed regions (i.e., colorable and colored with 2 colors) it is possible to associate with each vertex E a number $\sigma(E)=1$ or $\sigma(E)=2$ in such a way that in every region Γ the number of vertices with signature 1 or the number with 2 is always an even number (possibly zero) or always an odd number according as the region Γ belongs to the first or second class. Each vertex must be counted with its corresponding multiplicity. Theorem: The edges of an arbitrary net N of the sphere can be assigned the numbers 1 and 2 in such a way that at each vertex E the number of edges with signature 1 and likewise in each region Γ the number of bounding edges with signature 2 is always an even number, possibly zero.

III. The study of edge-transformations introduced in II is continued. An edge-transformation in a 3-net may be described roughly as follows. Any edge k is surrounded by four regions, two on opposite sides of k and two on the "ends" of k. The transformation replaces k by k' so as to reverse the roles of the pairs of regions mentioned. Definition: An even net is one in which the number of vertices in any region is even. Theorem: Any N_3 of the sphere, either regular or, more generally, satisfying the theorem of Petersen, may be edge-transformed to an even net by transforming a certain number of "non-passing" edges of a prescribed Petersen path of N_3.

IV. A certain theorem announced in III is proved. Definitions: If the number of edges belonging to any vertex is congruent to r mod t the net is said to be of vertex congruence class $[t, r]$ and denoted by $N_{t,r}$. If the number of edges of each region is congruent to ρ mod θ, the net is said to belong to the region congruence class (θ, ρ). If N belongs to both $[t, r]$ and (θ, ρ) it is designated by $N_{t,r}(\theta, \rho)$. An outer edge of a region Γ is an edge of the net having at least one vertex of Γ as an end point but which is not a bounding edge of Γ. A vertex transformation in a 3-net may be roughly described as one in which a vertex E is replaced by a "small" triangle Γ such that the three original edges of E are now three outer edges of Γ. Thus the number of regions is increased by one and the number of edges by three. The theorem in question then reads as follows. A 3-net N_3 of the sphere with exactly f regions can be transformed into a net $N_{3,0}(2, 1)$ by changing some vertices to triangles if and only if it is semi-even and contains one completely opposite system of regions with exactly g regions, where $g \not\equiv f$ mod 2. (We will not attempt to define here the terms "semi-even", or "completely opposite system of regions".)

L. M. Kelly (Berkeley, Calif.)

2988:

Hu, Sze-Tsen. On the uniqueness theorem of local homology theory. Tsing Hua J. Chinese Studies Special No. 1 (1959), 192–201. (Chinese summary)

This paper contains axioms for local homology theory and a proof of the uniqueness theorem for local homology functors which holds for the portions of categories containing locally triangulable pairs and continuous maps satisfying the condition: the inverse of the base point is a single point. The proof depends on the uniqueness theorem for homology of Eilenberg and Steenrod, and is shorter than the proof given by the reviewer [Duke Math. J. **25** (1958), 381–399; MR **20** #4264] for uniqueness of local homology functors on that portion of categories containing locally triangulable pairs and continuous maps.

T. R. Brahana (Athens, Ga.)

2989:

Nakaoka, Minoru. Decomposition theorem for homology groups of symmetric groups. Ann. of Math. (2) **71** (1960), 16–42.

Let $S(n)$ denote the symmetric group of degree n, let $\lambda=\lambda_n : S(n) \to S(n+1)$ be the natural inclusion, and $\lambda_* : H_*(S(n); G) \to H_*(S(n+1); G)$ the induced homology homomorphism, where G is a coefficient group on which both $S(n)$ and $S(n+1)$ operate trivially. Theorem: λ_* has a left inverse δ, i.e., $\delta\lambda_*=1$. Corollary:

$$H_*(S(n+1); G) \simeq H_*(S(n); G) \oplus H_*(S(n+1), S(n); G),$$

where the last term is the relative homology group.

The proof uses the following geometric description of $H_*(S(n))$, due to Steenrod: For every space X the group $S(n)$ operates on X^n ($=n$th cartesian power) by permutation of factors. If X is a simplicial complex and $\dim(X) \leq q$, then $S(n)$ operates freely (i.e., without fixed points) on $X^n - K$, where K is a subcomplex of dimension $\leq nq-q$. The integral simplicial cochains of X^n above dimensions $nq-q$ therefore form a free $S(n)$-complex, and if $X=S^q=$ q-sphere, q even, this complex is acyclic over the integers (Künneth formula). Hence

$$H_i(S(n); G) \simeq H^{nq-i}(\mathrm{SP}^n S^q; G), \quad i<q,\ q \text{ even},$$

where SP^n denotes the nth symmetric product. The author shows that under this isomorphism λ_* is transformed essentially into a cup-product, $\lambda_*=u\smile$, where

$$u \in H^q(\mathrm{SP}^{n+1}S^q; \mathbf{Z})$$

is the fundamental class. The inverse δ is now a certain combination of projection maps (which relate different symmetric products), slant products, and maps λ_{i*} with $i \leq n$.

Using his results on $H^*(\mathrm{SP}^n S^q; \mathbf{Z}_p)$ the author then determines $H_*(S(n+1), S(n); \mathbf{Z}_p)$. As a topological application he shows that for Eilenberg-MacLane groups $H^*(\mathbf{Z}, q; G)$ the cup-product $u\smile$ with the fundamental class $u \in H^q(\mathbf{Z}, q; \mathbf{Z})$ is a monomorphism. Finally he proves $H^k(\mathrm{SP}^n S^q) \simeq H^k(\mathrm{SP}^{n+1}S^q)$ for $k \leq 2n+q-2$.

All results have duals in cohomology [resp. homology].

A. Dold (New York)

2990:

Gugenheim, V. K. A. M. On a theorem of E. H. Brown. Illinois J. Math. 4 (1960), 292–311.

In Ann. of Math. (2) **69** (1959), 223–246 [MR **21** #4423], the reviewer has shown that the chains on a fibre space are chain equivalent to a chain complex which is the tensor product of the chains of the base and those of the fibre, the differential operator being the usual tensor product differentiation plus a cap product term. The present paper is devoted to stating and proving this result in the context of semi-simplicial complexes and twisted cartesian products. With this formulation the main result goes through smoothly. A generalization of a

theorem of Fadell and Hurewicz [ibid. **68** (1958), 314–347; MR **21** #2237a] is given as a corollary of the main theorem.
E. H. Brown (Waltham, Mass.)

2991:

Shih, Weishu. Détermination de la suite spectrale d'un espace fibré. C. R. Acad. Sci. Paris **250** (1960), 795–796.

The author considers a fibering E with base B, fibre F and structural group G. His intention [cf. Shih, same C. R. **248** (1959), 2935–2936; MR **21** #4424] is to give means for calculating the spectral sequence of E, starting from B, F and certain invariants of the fibering which can be stated in terms of B, F and G; more precisely, he uses a "twisting cochain" $f: C_*(B)\to C_*(G)$ [see Séminaire H. Cartan 1956/57, Exp. no. 4, Paris, 1958; MR **22** #1896c]. In terms of $C_*(B)$, $C_*(F)$, $C_*(G)$ and f he makes a fairly simple and elegant algebraic construction; he then states as a theorem that the result yields the spectral sequence of E, complete with its differentials. The formulae used in the construction are written in terms of cap products; the procedure thus provides a sort of generalisation of the Hurewicz-Fadell theorem [cf. #2990 above] which appears as a corollary.
J. F. Adams (Cambridge, England)

2992:

Fadell, Edward. The equivalence of fiber spaces and bundles. Bull. Amer. Math. Soc. **66** (1960), 50–53.

It is shown that every Hurewicz fibration $\pi:E\to X$ over a locally finite polyhedron X is fibre homotopy equivalent to a Steenrod bundle projection $\pi':E'\to X$. If S denotes the space of simplicial paths in X as defined by Milnor [Ann. of Math. (2) **63** (1956), 272–284; MR **17**, 994] then $E'=\{(e,\alpha)\in E\times S \mid \pi(e)=\alpha(0)\}$, and $\pi'(e,\alpha)=\alpha(1)$. A fibre-preserving map $f:E\to E'$ is obtained by $f(e)=(e,\pi(e))$. Using a lifting function for π (and the Milnor bundle) an explicit homotopy-inverse to $f|\pi^{-1}(X_0)$: $\pi^{-1}(X_0)\to\pi'^{-1}(X_0)$ is constructed, and the theorem follows from an earlier result of the author [Duke Math. J. **26** (1959), 699–706; MR **22** #233]. *A. Dold* (New York)

2993:

Halpern, Edward. The cohomology of a space on which an H-space operates. Michigan Math. J. **6** (1959), 381–394.

The author is concerned with the cohomology over a field K of characteristic zero of a space T on which an H-space X operates. This situation occurs repeatedly in the Postnikov decomposition of a space or fibre map. With some restrictions on X he proves that if $f:X\to T$ is a map which commutes (up to homotopy) with the operations of X on T and on itself, then $f^*H^*(T;K)$ is a Hopf-subalgebra of $H^*(X;K)$ and $H^*(T,K)\approx B\otimes C$, where B and C are subalgebras of $H^*(T,K)$ and f^* annihilates the elements of positive degree in B and is injective on C. For example in the Postnikov case T is fibred by X, which is an Eilenberg-MacLane space, and f can be taken to be the inclusion of a fibre in the total space.
N. Stein (New York)

2994:

Stewart, T. E. Lifting the action of a group in a fibre bundle. Bull. Amer. Math. Soc. **66** (1960), 129–132.

A space B is said to be a G-space for a topological group G if there is given a continuous map $\bar{a}:G\times B\to B$ such that $\bar{a}(u_1\cdot u_2,b)=\bar{a}(u_1,\bar{a}(u_2,b))$ and $\bar{a}(e,b)=b$. Let $\mathscr{B}$ be a bundle with base B and total space E. We say that the action $(G,\bar{a})$ can be lifted to E in $\mathscr{B}$ if E can be given the structure of a G-space so that the projection of E onto B in $\mathscr{B}$ is an equivariant map. The main theorem in this note states that, if G is a semi-simple, compact, connected Lie group, B is a paracompact space satisfying the first axiom of countability, and if $\mathscr{B}$ is a principal bundle with a torus H of dimension m as structural group, then the action $(G,\bar{a})$ can be lifted to E in $\mathscr{B}$.
S.-T. Hu (Los Angeles, Calif.)

2995:

Hermann, Robert. Obstruction theory for fibre spaces. Illinois J. Math. **4** (1960), 9–27.

In this paper, the author develops in detail the technique discussed in a previous announcement [Bull. Amer. Math. Soc. **65** (1959), 5–8; MR **22** #1903] for computing higher obstructions to cross-sections in a fiber-space. As illustrations of the method (for others, see previously mentioned note) the following are obtained: (1) a proof of Kundert's formula for the secondary obstruction where the fiber is a complex projective space [Kundert, Ann. of Math. (2) **54** (1951), 215–246; MR **13**, 374]; (2) a formula for the "secondary obstruction modulo a prime number l" in the case of a sphere-bundle; here a formula similar to Liao's appears [Liao, Ann. of Math. (2) **60** (1954), 146–191; MR **15**, 979]; (3) a formula concerning the difference of two secondary obstructions.
J.-P. Meyer (Baltimore, Md.)

2996:

Shimada, Nobuo. Triviality of the mod p Hopf invariants. Proc. Japan Acad. **36** (1960), 68–69.

One defines a homomorphism

$$H_p^{(k)}:\pi_{N+2p^k(p-1)-1}(S^N)\to Z_p$$

for each odd prime p and integer $k\geqq 0$ by attaching a cell of dimension $N+2p^k(p-1)$ to S^N by the homotopy class of maps α, computing the Steenrod operation $\wp^{p^k}$ in the resulting space X on the generator of $H^N(X;Z_p)$ and defining $H_p^{(k)}(\alpha)$ to be the coefficient of the generator of $H^{N+2p^k(p-1)}(X;Z_p)$. This is an analogue of the classical Hopf invariant. The author proves that $H_p^{(k)}$ is trivial whenever $k\geqq 1$, by decomposing $\wp^{p^k}$ in terms of secondary cohomology operations. *N. Stein* (New York)

2997:

Gutiérrez-Burzaco, Mario. Uniform homotopy groups. Nederl. Akad. Wetensch. Proc. Ser. A **63**=Indag. Math. **22** (1960), 67–73.

The author introduces, for uniform spaces Y, the notion of (relative) uniform homotopy groups $\pi_{n,p}(Y,A,a)$, $a\in A\subset Y$, depending on two indices n,p $(-1\leqq p\leqq n)$. The basic triple (I^n,I^{n-1},J^{n-1}), occurring in the definition of usual relative homotopy groups, is replaced by a new one. It consists of an n-cube with an open p-cell removed from the interior of its upper half $(I_n{}^p)$, of this upper half with the p-cell removed (I^{np}), and of the boundary of the original n-cube $(\dot{K}^n)$. $\pi_{n,p}$ consists of classes of uniformly homotopic uniform maps

$$f:(I_n{}^p,I^{np},\dot{K}^n)\to(Y,A,a).$$

For $p=-1$ and $p=n$, the groups $\pi_{n,p}$ reduce to the usual absolute and relative groups, respectively. Five types of homomorphisms connecting various homotopy groups are defined. The main theorem then establishes exactness of the homotopy sequence

$$\cdots \to \pi_{n+1,p+1}(Y,A) \to \pi_n(A) \to \pi_n(Y) \oplus \pi_{n,p}(A,A) \to \pi_{n,p}(Y,A) \to \cdots.$$

Finally, examples of spaces with non-trivial groups $\pi_{n,p}$ are exhibited. Clearly, the groups $\pi_{n,p}$ are invariants of uniform homotopy type.

S. Mardešić (Zagreb)

2998:

Weier, Josef. Invariante Schnittsysteme stetiger Transformationen. Math. Scand. **7** (1959), 33–48.

Let M and N be closed oriented triangulated manifolds of dimension m and n, respectively, and let $m=2n-2$, $n>2$. Let $f:M\to N$ be a map, a a point of N and $A=f^{-1}(a)$ an $(m-n)$-dimensional polyhedron with components $A_1, \cdots, A_t$. These components are classified by considering A_i equivalent to A_j whenever there is a path in M, leading from A_i to A_j, which maps by f into a zero-homotopic loop in N. To each of these classes the author assigns an $(m-n)$-cycle z_k, called a "homotopy component" of (f, a). z_k is defined following a pattern already exploited in several of his other papers [e.g.: Monatsh. Math. **62** (1958), 163–172; Collect. Math. **10** (1958), 45–58; MR **20** #4840; **21** #328]. To each pair (j, k) of integers not exceeding the number t of homotopy components, the author assigns an integer $\gamma(j,k)\in\{0, 1, -1\}$ and calls the function γ an "intersection system" (Schnittsystem). If $j\neq k$, $z_j=\partial x$, $z_k=\partial y$, and for any choice of $(n-1)$-chains x, y their intersection number is not zero, then $\gamma(j,k)\neq 0$. In all other cases $\gamma(j,k)=0$. The main theorem asserts invariance of $\gamma(j,k)$ under homotopy of f. It is also shown that γ is independent of the base point a.

S. Mardešić (Zagreb)

2999:

Schirmer, Helga. Bemerkungen zur Homotopietheorie der Koinzidenzen mehrerer Abbildungen. Arch. Math. **11** (1960), 56–64.

This note contains a discussion of the properties of coincidences of r mappings $(r\geq 2)$ $M\to N$, where M and N are manifolds, $\dim M=(r-1)\dim N$. The questions discussed include (1) can the mappings be deformed so as to eliminate the coincidences, (2) the homotopy invariance of an "algebraic coincidence number", (3) the existence of mappings with given coincidences and with minimum number of coincidences. The basic tool is the theory of obstructions to deformations, since the r mappings define a mapping $M\to N^r$ and (1), for example, is equivalent to deforming this mapping away from the diagonal. The problem here is considerably simplified, since the assumptions made imply that only the first obstruction enters into the problem.

An earlier work of the author [J. Reine Angew. Math. **194** (1955), 21–39; MR **17**, 394] considers the case $n=2$.

J.-P. Meyer (Baltimore, Md.)

3000:

Wu, Wen-tsün. On the realization of complexes in Euclidean spaces. II. Sci. Sinica **7** (1958), 365–387.

English version of Acta Math. Sinica **7** (1957), 79–101 [MR **20** #3536].

DIFFERENTIAL TOPOLOGY

3001:

Peetre, Jaak. Une caractérisation abstraite des opérateurs différentiels. Math. Scand. **7** (1959), 211–218.

X being a real indefinitely differentiable manifold, let $\tilde{\mathscr{E}}$ be the stack (=sheaf, faisceau) of germs of C^∞ functions on X. The author shows that any homomorphism P of $\tilde{\mathscr{E}}$ into itself is a linear partial differential operator in the following sense. For each point x of X there exists a coordinate system $x_1, \cdots, x_n$ with domain a neighbourhood Ω of x and a locally finite family of functions a^α, each indefinitely differentiable on Ω, such that P is represented throughout Ω by $\sum_\alpha a^\alpha(\partial/\partial x)_\alpha$; herein each α is a finite (possibly empty) sequence of natural numbers lying between 1 and n (=dim X), say $\alpha=(\alpha_1, \cdots, \alpha_p)$, and $(\partial/\partial x)_\alpha=\partial^p/\partial x_{\alpha_1}\cdots\partial x_{\alpha_p}$. Since there is an obvious converse, we have here the abstract characterisation referred to in the title.

R. E. Edwards (Reading)

3002:

Weier, Joseph. Über gewisse Integralinvarianten bei der Transformation von Differentialformen. Rend. Circ. Mat. Palermo (2) **8** (1959), 311–332.

Let M and N be compact differentiable manifolds of dimensions m and n, respectively, and let $f:M\to N$ be a differentiable mapping. Let f^* be the linear map induced by f of the differential forms on N into the differential forms on M. If $m=2n-1$, if w is an n-form on N with $\int_N w=1$ and if $f^*w=d\alpha$, then the integral $\int_M \alpha\wedge f^*w$ depends only on the homotopy class of f (i.e., if f' is homotopic to f and if $(f')^*w'=d\alpha'$ with $\int_N w'=1$, then $\int_M \alpha'\wedge(f')^*w'=\int_M \alpha\wedge f^*w$). The author proves several theorems which generalize the above. For example, if the degree of w is r, if $m=2r-1$ and if $r\leq n<2r-1$, then $\int_M \alpha\wedge f^*w$ (for fixed f) depends only on the cohomology class of w.

The author also proves the following theorem. Assume that $n\geq 2$ and $m=3n-2$, M and N are orientable, there exists some point $y\in N$ such that $f^{-1}(y)$ is a $(2n-2)$-dimensional submanifold of M, and w is an n-form on N with $\int_N w=1$ and $f^*w=d\alpha$. Then

$$\int_M \alpha\wedge\alpha\wedge f^*w=\int_{f^{-1}(y)}\alpha\wedge\alpha.$$

Under the same hypothesis as above the author obtains another result. Let $A_1, A_2, \cdots$ be equivalence classes of $(n-1)$-forms, where α is equivalent to α' if $\alpha-\alpha'$ is exact and if $d\alpha=d\alpha'=f^*w$. For each $\alpha_i\in A_i$ let $x_i=\int_M \alpha_i\wedge\alpha_i\wedge f^*w$. Then the x_i are uniquely determined by the equivalence classes A_i.

J. J. Kohn (Waltham, Mass.)

3003:

Eells, James, Jr. On submanifolds of certain function spaces. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1520–1522.

Given a C^∞ manifold M and a regularly imbedded closed submanifold B, let E_B denote the function space of paths (maps of [0, 1], absolutely continuous in the metric of M and having square integrable tangent maps), starting at a fixed point $m_0\in M$ and ending on B. Also consider E_A, where A is a submanifold of B of codimension

$r>0$. Theorem: For each dimension i there is a canonical isomorphism

$$\theta: H^{i-r}(E_A; \mathfrak{F}) \to H^i(E_B, E_{B-A}).$$

Cohomology groups are singular with coefficients from a certain local system $\mathfrak{F}$ of integers and with integer coefficients, respectively. The definition of $\mathfrak{F}$ and the proof rely on the fact that E_A and E_B are infinite-dimensional Riemann manifolds modeled on a separable Hilbert space [J. Eells, Jr., *Symposium international de topología algebraica*, pp. 303–308, Univ. Nac. Aut. México, Mexico City, 1958; MR **20** #4878]. *S. Mardešić* (Zagreb)

3004:

Smale, Stephen. Diffeomorphisms of the 2-sphere. Proc. Amer. Math. Soc. **10** (1959), 621–626.

An analogue for differentiable homeomorphisms is established for a classical theorem of Kneser concerning homeomorphisms on the unit sphere S^2 in Euclidean 3-space. It is shown that if Ω is the space of all orientation-preserving C^∞ diffeomorphisms of S^2 with the C^r topology as described by Thom, then the rotation group SO(3) is a strong deformation retract of Ω. A diffeomorphism is a differentiable homeomorphism with a differentiable inverse; and the mapping involved in the retraction homotopy are themselves C^∞ differentiable.

G. T. Whyburn (Charlottesville, Va.)

3005:

Munkres, James. Obstructions to the smoothing of piecewise-differentiable homeomorphisms. Bull. Amer. Math. Soc. **65** (1959), 332–334.

L'auteur annonce une mise en ordre au moyen de la théorie des obstructions de résultats de Milnor et Thom concernant la classification des structures différentiables des variétés. (Il s'agit toujours de structures différentiables combinatoirement équivalentes, c'est-à-dire possédant des C^2-triangulations isomorphes.) Soient M et N deux variétés de dimension n munies de C^2-triangulations et soit f un isomorphisme $M \to N$ linéaire au sens de ces triangulations. On cherche à modifier "légèrement" f, en descendant sur le squelette, de manière à la rendre différentiable sur les simplexes ouverts de dimension $n-1$, puis $n-2$, etc. Supposons l'opération réussie jusqu'au m-squelette et soit g l'application déformée. On définit alors une classe d'homologie $\lambda_m g \in H_m(M, \Gamma_{n-m})$ où Γ_p est le groupe (abélien) quotient du groupe Dif (S^{p-1}) des difféomorphismes positifs de la sphère S^{p-1} par le sous-groupe Dif (B^p) de ceux qui s'étendent à la boule. Si $\lambda_m g = 0$ on peut poursuivre l'opération désirée jusqu'au $(m-1)$-squelette. Cette méthode permet, entre autres, de redémontrer des résultats connus sur les structures différentiables de R^n et S^n [Thom, Proc. Internat. Congr. Math. (Edinburgh 1958), pp. 248–255, Cambridge University Press, New York, 1960; Colloque Géom. Diff. Globale (Bruxelles, 1958), pp. 27–35, Centre Belge Rech. Math., Louvain, 1959]. *P. Dedecker* (Liège)

3006:

Hermann, Robert. A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle. Proc. Amer. Math. Soc. **11** (1960), 236–242.

Let X and B be Riemannian manifolds, ϕ a mapping from X onto B, $\phi_*: X_x \to B_{\phi(x)}$ the differential of ϕ, where X_x is the tangent space to X at x. Main theorem: Suppose that $\phi_*(X_x) = B_{\phi(x)}$ for all x in X, that the isomorphism $\phi_*: X_x/\phi_*^{-1}(0) \approx B_{\phi(x)}$ preserves the inner products and that X is complete. Then B is complete and $\phi: X \to B$ is a locally trivial fibre space. If, in addition, the fibres are all totally geodesic submanifolds, then $\phi: X \to B$ is a fibre bundle with structure group the group of isometries of the fibre. The proof is based on the following proposition. The projection of a horizontal geodesic of X is a geodesic of B and, conversely, the horizontal lifting of a geodesic of B is a geodesic of X. (The orthogonal complement of $\phi_*^{-1}(0)$ is called horizontal.) An example generalizing the situation where X is the bundle of orthonormal frames of B is discussed. *S. Kobayashi* (Vancouver, B.C.)

3007:

Sasaki, Shigeo. On the differential geometry of tangent bundles of Riemannian manifolds. Tôhoku Math. J. (2) **10** (1958), 338–354.

The author derives some relations between an n-dimensional differentiable manifold M^n and the total space of its tangent bundle $T(M^n)$, using the tools of classical tensor analysis; these relations, mostly of local character, are obtained from certain homogeneous homomorphisms of the additive structure of the sheaf $\mathfrak{A}(M^n)$ [reviewer's notation] of germs of differentiable cross sections in the tensor algebra over M^n into the corresponding sheaf $\mathfrak{A}(T(M^n))$. One assumes known the natural ring homomorphism of $\mathfrak{A}(M^n)$ into $\mathfrak{A}(T(M^n))$ generated by the natural injection of scalars and covariant tensors from M^n to $T(M^n)$ and by interpreting each tangent vector ξ_x at any point $x \in M^n$ as an infinitesimal translation of the tangent space $T_x \subset T(M^n)$ at x into itself. Besides this, the author defines another mapping, which turns out to be a derivation of $\mathfrak{A}(M^n)$ into $\mathfrak{A}(T(M^n))$, called the extension, denoted by a bar placed over any given germ of a tensor over M^n. This map may be characterized as follows. For any germ of a scalar f at $x \in M^n$ of class C^1 the extended scalar $\bar{f}$ in $T(M^n)$ is the one which at the point $v_x \in T_x \subset T(M^n)$ takes the value $\bar{f}(v_x) = v_x(f)$; for any germ ξ of a differentiable vector field at $x \in M^n$ the extended field $\bar{\xi}$ in $T(M^n)$ is a section in the sheaf over T_x, and for each $v_x \in T_x$ the vector $\bar{\xi}(v_x)$ has $\xi(x)$ as its natural projection in M^n and depends linearly on v_x; for any germ η of a covariant vector field at $x \in M^n$ the extended germ $\bar{\eta}$ is defined over T_x, and its restriction to the tangent space of T_x has the same effect as η. For any germ ξ of a tangent vector field at x, one verifies the identity $\bar{\eta}(\bar{\xi}) = \overline{\eta(\xi)}$. Independently of the above construction, given a Riemannian structure in M^n one can define a natural Riemannian structure on $T(M^n)$, characterized by the properties that the fibres are totally geodesic, mutually parallel, Euclidean vector spaces with the norm defined by the Riemannian form, the infinitesimal distance between fibres equals that between their projections in M^n, and any differentiable curve $\bar{C}$ in $T(M^n)$ that is an orthogonal trajectory of the fibres corresponds to a Levi-Civita parallelism of a tangent vector along the projection C of $\bar{C}$ in M^n.

With the aid of these concepts a number of results (6 lemmas, 18 theorems, 5 corollaries) are stated and proved. Here are samples. Theorem 8: Let M^n be a Riemannian manifold whose Ricci tensor is equal to zero. Then the

extended vector field to a covariant, harmonic vector field ξ in M^n is harmonic in $T(M^n)$ as well. The equations of geodesics in $T(M^n)$ with respect to its Riemannian structure are derived; their projections in M^n are called submarine geodesics. Theorem 16: If M^n is a Riemannian manifold of constant curvature $K\neq 0$, then every submarine geodesic has third curvature identically zero.

The article eschews modern notations and terminology; the resulting greater length of exposition is compensated by its easy accessibility to readers with a classical background. *E. Calabi* (Minneapolis, Minn.)

3008:

Yamaguchi, Satoru. On complex parallelisable manifolds. Mem. Fac. Sci. Kyushu Univ. Ser. A. **13** (1959), 136–139.

It is proved that the fundamental group $\pi_1(M)$ of a compact parallelisable manifold M cannot be finite. In particular, when M admits a Kählerian structure, $\pi_1(M)$ is infinitely cyclic. It is also shown that compact connected real Lie groups are complex parallelisable manifolds if and only if they are even-dimensional tori. The author makes use of the result [cf. S. Bochner and W. T. Martin, *Several complex variables*, Princeton Univ. Press, 1948; MR **10**, 366] that compact connected complex Lie groups are all abelian. *T. J. Willmore* (Liverpool)

3009:

Kodaira, K.; Spencer, D. C. On deformations of complex analytic structures. I, II. Ann. of Math. (2) **67** (1958), 328–466.

The reviewer feels that he cannot describe the contents of this long and important paper better than by quoting from the introduction.

"The purpose of the present paper is to develop a more or less systematic theory of deformations of complex structures of higher dimensional manifolds. The concept of deformations of complex structures, or of a family of complex structures depending differentiably on a parameter, can be defined in terms of structure tensors determining the complex structures [see Kodaira and Spencer, *Algebraic geometry and topology*, pp. 139–150, Princeton Univ. Press, 1957; MR **19**, 578; Frölicher and Nijenhuis, Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 239–241; MR **18**, 762]. Given a family $\{V_t\}$ of complex structures V_t defined on a compact differentiable manifold X which depend differentiably on a parameter t moving on a connected differentiable manifold M, the union $\mathscr{V}=\bigcup_{t\in M} V_t$ of the complex manifolds V_t may be regarded as a kind of fibre space over M whose structure is a mixture of differentiable structure and complex structure along the fibres. Reversing this process, we define a differentiable family of compact structures (manifolds) as a fibre space $\mathscr{V}$ over a connected differentiable manifold M whose structure is a mixture of differentiable and complex structures. In particular, if $\mathscr{V}$ and M are both complex manifolds and if $\mathscr{V}$ is a complex-analytic fibre space over M, we call $\mathscr{V}$ a complex analytic family of compact complex structures (manifolds).

"Our first task is to define an object which measures the magnitude of dependence of the complex structure of V_t on the parameter t. We introduce a sheaf Θ on $\mathscr{V}$, the corresponding sheaf of cohomology $\mathscr{H}^1(\Theta)$ on M, and construct a homomorphism $\rho: T_M\to\mathscr{H}^1(\Theta)$ of the sheaf T_M of germs of differentiable vector fields on M into $\mathscr{H}^1(\Theta)$. This homomorphism ρ may be considered to be the object which represents the magnitude of dependence of the complex structure of V_t on t. In fact, $\mathscr{V}$ is locally trivial in the sense that $\mathscr{V}$ has a local product structure (and therefore the complex structure of V_t is independent of t) if and only if ρ vanishes. The restriction Θ_t of Θ to a fibre V_t of $\mathscr{V}$ coincides with the sheaf of germs of holomorphic vector fields on V_t and therefore $H^1(V_t,\Theta_t)$ depends only on V_t. By restricting ρ to V_t we obtain a homomorphism $\rho_t:(T_M)_t\to H^1(V_t,\Theta_t)$ which was first introduced by Frölicher and Nijenhuis [loc. cit.], where $(T_M)_t$ denotes the tangent space of M at t. For any tangent vector $v_t\in(T_M)_t$, the image $\rho_t(V_t)\in H^1(V_t,\Theta_t)$ represents the 'infinitesimal deformation' of the complex structure of V_t along the vector v_t. Clearly $\rho=0$ implies $\rho_t=0$ for all $t\in M$, but the converse is not necessarily true, as an example shows. This is connected with the fact that the complex structure can 'jump' from one structure to another by an arbitrarily small deformation, a phenomenon which does not occur in the case of Riemann surfaces. Because of this phenomenon, it is impossible to generalize the concept of distance between two compact Riemann surfaces in the sense of Teichmüller [Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1939**, no. 22; MR **2**, 187] to higher dimensional manifolds. With the help of the theory of harmonic forms we prove that, if $\dim H^1(V_t,\Theta_t)$ is independent of t, $\rho_t=0$ for all $t\in M$ implies $\rho=0$, and derive from this the theorem of Frölicher and Nijenhuis [loc. cit.] concerning the rigidity of compact complex manifolds.

"In case $\mathscr{V}$ is a complex analytic family, i.e., a complex analytic fibre space, we may ask whether $\mathscr{V}$ is a complex analytic fibre bundle if $\mathscr{V}$ is locally trivial as a differentiable family. We give an affirmative answer to this question.

"Each imbedding of a compact complex manifold V_o as fibre over the point $o\in M$ in a differentiable family $\mathscr{V}\to M$ of complex structures determines a space of infinitesimal deformations in $H^1(V_o,\Theta_o)$, namely the image of the map $\rho_o:(T_M)_o\to H^1(V_o,\Theta_o)$. A space of infinitesimal deformations in $H^1(V_o,\Theta_o)$ determined in this way will be called maximal if it is not a proper subspace of a space of infinitesimal deformations determined by some other imbedding of V_o as fibre in a differentiable family. A maximal space of infinitesimal deformations will be called a deformation space. An example shows that there exist manifolds V_o with more than one deformation space in $H^1(V_o,\Theta_o)$, at least for complex dimension exceeding 2.

"We generalize the above results to differentiable families of complex manifolds with the additional structure of complex fibre bundle.

"Next, we extend Riemann's concept of number of moduli to higher dimensional complex manifolds. The main point here is to avoid the use of the concept of the space of moduli of complex manifolds which cannot be defined in general for higher dimensional manifolds. Moreover, a necessary condition for the existence of a number $m(V_o)$ of moduli of a complex manifold V_o is that $H^1(V_o,\Theta_o)$ contain only one deformation space; hence $m(V_o)$ is not defined for all compact complex manifolds. We compute the number $m(V_o)$ of moduli of some simple types of complex manifolds V_o and find that $m(V_o)=\dim H^1(V_o,\Theta_o)$. We remark that, in the case of a

Riemann surface V_o, the duality theorem implies that $H^1(V_o, \Theta_o)$ is isomorphic to the space of quadratic differentials which are everywhere regular on V_o; therefore $m(V_o) = \dim H^1(V_o, \Theta_o)$ coincides with the number of moduli of V_o in the sense of Riemann. On the other hand, the classical formula for the number of moduli of algebraic surfaces due to Noether [S.-B. Preuss. Akad. Wiss. Berlin **1888**, Halbband 1, 123–127] may be regarded as the 'postulation formula' for $\dim H^1(V_o, \Theta_o)$. In view of the above statements, we would like to propose, as the main problem in the theory of deformations of compact complex manifolds, that of understanding the reasons why the equality $m(V_o) = \dim H^1(V_o, \Theta_o)$ holds for many examples of complex manifolds.

"By restricting our considerations to submanifolds of a fixed compact complex manifold W, we obtain the corresponding theory of deformations of complex structures relative to W. Thus we introduce the number $m_W(V_o)$ of relative moduli and the 'relative cohomology' $H_W{}^1(V_o, \Theta_o)$ for a submanifold V_o of W. In the particular case in which W is an algebraic manifold and V_o is a submanifold of W of codimension 1, we derive the equality $m_W(V_o) = \dim H_W{}^1(V_o, \Theta_o)$ from the theorem of completeness of the characteristic linear systems of complete continuous systems."

M. F. Atiyah (Cambridge, England)

3010:

Kodaira, K.; Spencer, D. C. A theorem of completeness for complex analytic fibre spaces. Acta Math. **100** (1958), 281–294.

Let $\mathscr{V} \to M$ be a complex analytic family of compact complex manifolds. In the notation of #3009, the authors prove the following theorem: if, for some $t \in M$, ρ_t is surjective, then $\mathscr{V} \to M$ is complex analytically complete at t. The proof is elementary, in the sense that no use is made of the theory of harmonic forms. Many examples of complete families of complex manifolds fall within the scope of this theorem.

M. F. Atiyah (Cambridge, England)

3011:

Kodaira, K.; Spencer, D. C. A theorem of completeness of characteristic systems of complete continuous systems. Amer. J. Math. **81** (1959), 477–500.

The authors prove the following theorem. Let W be a (complex projective) algebraic manifold, V_0 a submanifold of W of codimension 1. Suppose V_0 is semi-regular; then V_0 belongs to one and only one complete continuous system $\mathscr{S}$ of effective divisors on W, and V_0 corresponds to a simple point of the canonical parameter variety of $\mathscr{S}$. Moreover the characteristic system of $\mathscr{S}$ on V_0 is complete.

In the statement of this theorem the crux of the matter lies in the last sentence, the preceding facts being necessary in order that the notion of characteristic system be defined. The condition "V_0 is semi-regular" means that the canonical system of W cuts on V_0 a complete linear system.

In the form just given this theorem includes all earlier versions. The proof moreover is elementary and requires a minimum of assumptions. In fact the argument is local in the neighbourhood of V_0 in W and so W need not be algebraic or compact. The general method of proof is as follows. The problem is first formulated in cohomological terms, using the general notions of families of complex manifolds developed by the authors [#3009]. In this form the essential problem is the construction of a certain family of complex manifolds V_t which are deformations of V_0. First it is shown that the condition of semi-regularity on V_0 is precisely what is required to give a "formal" family. The proof is then completed by a delicate argument showing that the formal solution converges.

M. F. Atiyah (Cambridge, England)

3012:

Kodaira, K.; Nirenberg, L.; Spencer, D. C. On the existence of deformations of complex analytic structures. Ann. of Math. (2) **68** (1958), 450–459.

In this paper the authors prove the following existence theorem [the terminology is that of #3009; see also #3010]. Let V_0 be a compact complex manifold and let Θ_0 be the sheaf over V_0 of germs of holomorphic vector fields. If $H^2(V_0, \Theta_0) = 0$, then there exists a complex analytic family $\mathscr{V} = \{V_t | t \in M\}$ of deformations V_t of V_0 such that ρ_0 maps the tangent space $(T_M)_0$ of M at 0 isomorphically onto $H^1(V_0, \Theta_0)$.

The proof is based on the theory of partial differential equations.

M. F. Atiyah (Cambridge, England)

3013:

Kodaira, K.; Spencer, D. C. Existence of complex structure on a differentiable family of deformations of compact complex manifolds. Ann. of Math. (2) **70** (1959), 145–166.

The authors consider the problem of giving a differentiable family of compact complex manifolds the structure of a complex analytic family $\mathscr{V} \xrightarrow{\varpi} M$ [see #3009 for terminology and notation]. For $\mathscr{V}$ to be complex analytic it is trivially necessary that: (i) the image of ρ_t is a complex submodule of $H^1(V_t, \Theta_t)$. One of the main results of this paper can then be stated as follows. If, in addition to (i), we have: (ii) $\dim H^1(V_t, \Theta_t)$ is independent of t, (iii) ρ_t is injective for all $t \in M$, then each point of M has a neighbourhood U such that $\varpi^{-1}(U) \to U$ has the structure of a complex analytic family (compatible with the given structures).

The existence of a global complex structure, i.e. with $U = M$, is shown to be false in general. If, however, each V_t has no continuous group of automorphisms, then the complex structure of $\varpi^{-1}(U)$ is unique, from which it follows that there is a global structure. The uniqueness of the complex structure of M is proved in all cases.

The theorems are proved by exhibiting almost complex structures which are integrable, and then applying the fundamental theorem of Newlander and Nirenberg [Ann. of Math. (2) **65** (1957), 391–404; MR **19**, 577] on the existence of complex structures.

M. F. Atiyah (Cambridge, England)

3014:

Kobayashi, Shoshichi. On the automorphism group of a certain class of algebraic manifolds. Tôhoku Math. J. (2) **11** (1959), 184–190.

The author proves the following theorem. Let M be a compact complex manifold whose first Chern class $c_1(M)$

is negative definite. Then the group G of all holomorphic transformations of M is finite. This improves a result of Nakano [cf. #3009] to the effect that G is discrete, but the present proof is independent of Nakano's proof and is very simple. First, using the basic result of Kodaira [Ann. of Math. (2) **60** (1954), 28–48; MR **16**, 952], one knows that for some integer $k>0$ the line-bundle kK (where K is the canonical line-bundle) will define a projective embedding $M \to P_N$, where P_N is the projective space associated to the dual of the vector space $\Gamma(kK)$ of holomorphic cross-sections of kK. It is easy to see that G is then represented by a closed complex analytic subgroup of the group of projective transformations of P_N. On the other hand the author shows that G leaves invariant a bounded domain in the space $\Gamma(kK)$. Since G keeps the origin fixed it follows, by a theorem of H. Cartan [*Sur les groupes de transformations analytiques*, Actualités Sci. Ind. No. 198, Hermann, Paris, 1935], that G is compact. Thus G is a compact complex analytic subgroup of the projective group and so is finite.

There is an application to fibre bundles with fibre M, giving conditions under which the total space is Kählerian or algebraic. *M. F. Atiyah* (Cambridge, England)

3015:

Boothby, W. M.; Wang, H. C. On contact manifolds. Ann. of Math. (2) **68** (1958), 721–734.

A Pfaffian form ω on a $(2n+1)$-dimensional differentiable manifold M is called a contact form if $\omega \wedge (d\omega)^n \neq 0$ everywhere on M; cf. the opening remarks to review #3016 below. The vector field Z on M characterized by the following two properties is said to be associated to ω: (1) $\omega(Z)=1$ and (2) $\iota(Z)\cdot d\omega=0$ ((2) can be replaced by $L(Z)\omega=0$, where $L(Z)$ is the Lie derivation with respect to Z). The contact form ω is said to be regular if each point of M has a cubical coordinate neighborhood $(x^1, \cdots, x^{2n+1})$ whose intersection with any integral curve defined by Z is given by $x^1=c_1, \cdots, x^{2n}=c_{2n}$, c_i constant. If ω is regular and M is compact, then each integral curve defined by Z is a simple closed curve and, hence, a positive function λ on M can be defined by

$$\lambda(p) = \inf\{t>0;\ \exp(tZ)\cdot p = p\}, \quad p \in M.$$

It is shown that λ is differentiable. In part I of this paper, the authors characterize compact regular contact manifolds by proving the following theorems.

Theorem 1: If ω is regular and M is compact, the vector field Z' associated to the contact form $\omega'=(1/\lambda)\omega$ generates the circle group S^1 acting effectively and freely (i.e., no element of S^1, except the identity, leaves a point fixed) on M.

Theorem 2: In the notation of theorem 1, let $B=M/S^1$; then (i) B is a differentiable manifold and M is a principal fibre bundle over B with group S^1; (ii) ω' defines a connection in this circle bundle M, (iii) the curvature form $d\omega'$ of this connection, projected onto the base space B, defines a symplectic structure on B. Let Ω be the 2-form defining the symplectic structure on B, i.e., $\pi^*(\Omega)=d\omega'$, where π is the projection of M onto B. Ω being the characteristic class of the circle bundle M, it is integral.

Theorem 3 (converse of theorem 2): Given a symplectic manifold B whose fundamental 2-form Ω is integral, let M be a principal circle bundle over B and ω a connection in the bundle M such that $d\omega=\pi^*(\Omega)$. (Such a pair (M, ω) always exists.) Then ω is a regular contact form whose associated vector field generates the right translation of the bundle M by S^1.

Let M be a coset space of a connected Lie group G by a closed subgroup K, i.e., $M=G/K$. A contact form ω on M is said to be homogeneous if ω is invariant by G. In part II the authors study a homogeneous contact form. First of all, they prove that a homogeneous contact form is always regular (theorem 4). More complete results are obtained under additional assumptions; theorem 6: If $M=G/K$ is a compact simply connected homogeneous contact manifold, then M is a principal fibre bundle over a Kaehlerian coset space B of a compact semi-simple Lie group (B is rational algebraic with an Einstein-Kaehler metric).

Let $\omega^*=p^*(\omega)$ where $p: G \to G/K=M$ is the natural projection. Let $H=\{h \in G;\ \text{ad.}\ h(\omega^*)=\omega^*\}$. Then, under the same hypothesis as in theorem 6, the authors prove theorem 7: The bundle M over B with group S^1 is equivalent to the bundle G/K over G/H with group H/K. They prove also that if a semi-simple Lie group G admits a left invariant contact form, then G is of rank 1, hence locally isomorphic to either SO(3) or SL(R, 2) (theorem 5).

In part III, the authors show that the converse of theorem 6 (a corollary of theorem 3) furnishes new examples of contact manifolds; in fact, they prove, by an algebraic topological argument, that many of the circle bundles M (constructed in theorem 3) over Kaehlerian coset spaces of compact semi-simple Lie groups are neither cotangent sphere bundles nor spheres.

{Remark by the reviewer. The proof of theorem 1 is incomplete. In order to prove that $Z'=\lambda Z$ is the vector field associated to ω', it is necessary to show not only that λ is differentiable but that λ is constant. The authors have communicated a proof of this fact to the reviewer.}

S. Kobayashi (Cambridge, Mass.)

3016:

Gray, John W. Some global properties of contact structures. Ann. of Math. (2) **69** (1959), 421–450.

A $(2n+1)$-dimensional manifold M is a contact manifold if there exist an open covering $\{U_i\}$ of M and a 1-form α_i on each U_i such that (1) $\alpha_i \wedge (d\alpha_i)^n \neq 0$ everywhere on U_i and (2) if $U_i \cap U_j \neq \emptyset$ then $\alpha_i=\sigma_{ij}\alpha_j$, where σ_{ij} is a function on $U_i \cap U_j$. The element of $H^1(M, Z_2)$ defined by {sign σ_{ij}} is the 1st Stiefel-Whitney class of M. Hence, if M is orientable, there exists a 1-form α (called a contact form) on M such that (1) $\alpha \wedge (d\alpha)^n \neq 0$ everywhere on M and (2) $\alpha_i=\sigma_i\alpha$ on U_i, where σ_i is a function on U_i. The structure group of the tangent bundle of an orientable contact manifold can be reduced to $U(n)\times 1$, hence the odd-dimensional Stiefel-Whitney classes of M are all zero [S. S. Chern, Colloque Internat. Strasbourg 1953, pp. 119–136, C.N.R.S., Paris, 1953; MR **16**, 112]. Using this result of Chern, the author proves that if M is a compact orientable contact manifold with a contact form α, then M is the boundary of a (not necessarily compact) manifold N with boundary which carries a 1-form β such that $(d\beta)^{n+1} \neq 0$ and $\beta|M=\alpha$. The proof seems unnecessarily complicated; the essential point of his proof is the following. Set $N=M\times\{0<t\leq 1\}$ and $\beta=t\cdot\pi^*(\alpha)$, where $\pi: N \to M$ is the natural projection. (Identify M with $M\times\{1\}$). This reasoning does not require the compactness assumption of M. (Note that the author is not claiming N to be compact.) In addition to the classical examples (i.e., the

cotangent sphere bundles, the odd-dimensional spheres), new examples (a certain class of hypersurfaces in a Euclidean space, a non-orientable contact manifold) are discussed. The author concludes Chap. I by studying an almost contact manifold (i.e., a manifold whose structure group can be reduced to $U(n)\times 1$). It is proved that, for an orientable manifold M, the primary obstruction to the existence of an almost contact structure is $W_3(M)$ and that a 5-dimensional orientable manifold M is almost contact if and only if $W_3(M)=0$.

A vector field v on a contact manifold M with a contact form α is an infinitesimal contact transformation (i.c.t.) if $\lambda_\alpha(v)=\sigma v$, where $\lambda_\alpha(v)$ denotes the Lie derivative of α with respect to v and σ is a function on M. If $\sigma=0$, then v is called a strict i.c.t. (s.i.c.t.). In Chap. II the author studies these transformations. Let Θ_α [resp. $\tilde{\Theta}_\alpha$ and A_α] be the sheaf of germs of i.c.t. [resp. s.i.c.t. and C^∞ multiples of α]. He proves that

$$0\to H^0(M,\tilde{\Theta}_\alpha)\to H^0(M,\Theta_\alpha)\to H^0(M,A_\alpha)$$
$$\to H^1(M,\tilde{\Theta}_\alpha)\to 0$$

is exact and that $H^q(M,\tilde{\Theta}_\alpha)=0$ for $q\geqq 2$.

The last section of the paper is devoted to the study of the deformations of a contact structure based on the method originated by Kodaira and Spencer [#3009]. The first theorem states that if α_t, $|t|<1$, is a 1-parameter family of contact forms on M, then there exists a 1-parameter family of transformations f_t, $|t|<\varepsilon$ for some ε, of M such that $f_t^*(\alpha_t)=\tau_t\alpha_0$, where τ_t is a function different from zero everywhere. Let $\{U_i\}$ be an open covering of M. Given a 1-parameter family of contact forms α_t, let $(x_i^1(t),\cdots,x_i^n(t),y_i^1(t),\cdots,y_i^n(t),z_i(t))$ be a canonical coordinate system in U_i, differentiable in t:

$$\alpha_t|U_i=dz_i(t)-\sum_k y_i^k(t)dx_i^k(t).$$

Let $h_{ij}(x_j,y_j,z_j;t)$ be the Jacobian matrix $\partial(x_i(t),y_i(t),z_i(t))/\partial(x_j(t),y_j(t),z_j(t))$. Then $(\partial h_{ij}/\partial t)_{t=0}$ is a s.i.c.t. on $U_i\cap U_j$. Hence, $(\partial h_{ij}/\partial t)_{t=0}$ defines an element of $H^1(M,\Theta_{\alpha_0})$. The 2nd theorem states that, conversely, every element of $H^1(M,\tilde{\Theta}_\alpha)$ comes from some 1-parameter family of contact forms α_t such that $\alpha_0=\alpha$. The deformations of a contact form whose associated vector field defines a fibration of M by a circle are discussed. The structure of such a contact manifold has been completely determined by Boothby and Wang [#3015].

S. Kobayashi (Cambridge, Mass.)

3017:

Kobayashi, Shoshichi. Geometry of bounded domains. Trans. Amer. Math. Soc. **92** (1959), 267–290.

This paper deals with the study of differential geometric properties of bounded domains in $\mathbf{C}^n$. The author considers on an n-dimensional complex manifold M the Hilbert space F of the square integrable holomorphic n-forms. By means of an orthonormal basis of F the kernel (n,n)-form (i.e., a C^∞ density) K on M is constructed, which does not depend on the choice of the orthonormal basis in F. The local components of this density can be assumed as local potentials of a Kähler metric—the Bergman metric—in M, if the following condition A is satisfied: Given any point $z\in M$ and any complex tangent vector Z at z, there exist two square integrable holomorphic n-forms f and f_1 such that $f(z)\neq 0$, $f_1(z)=0$ and $Z(f_1^*)\neq 0$, where $f_1=f_1^*dz^1\wedge\cdots\wedge dz^n$ is the local representation of f_1 in a neighborhood of z. Examples of manifolds satisfying condition A are given by the bounded domains in a Stein manifold and by the nonsingular hypersurfaces of degree $>n+2$ in $P_{n+1}(\mathbf{C})$.

From now on we shall always assume that condition A is satisfied. Then the group $G(M)$ of all holomorphic transformations of M (which leave invariant the Bergman metric of M) is a Lie group. Moreover, the isotropy subgroup of $G(M)$ at every point of M is compact. This generalizes a well-known theorem of H. Cartan for bounded domains [*Sur les groupes de transformations analytiques*, Actualités Sci. Ind. No. 198, Hermann, Paris, 1935]. Let $g_{\alpha\beta}dz^\alpha\overline{dz^\beta}$ and $R_{\alpha\beta}dz^\alpha\overline{dz^\beta}$ denote respectively the Bergman metric and its Ricci form. It is proven then that the holomorphic sectional curvature of M is less than or equal to 2 and that $[(n+1)g_{\alpha\beta}-R_{\alpha\beta}]dz^\alpha\overline{dz^\beta}$ is a positive semidefinite hermitian form. (For a bounded domain in $\mathbf{C}^n$ these statements are true in the strong sense.) M is complete with respect to the Bergman metric if the following condition B is satisfied: For every infinite sequence S of points of M with no adherent point in M, and for each $f\in F$ there exists a subsequence S' of S such that $\lim_{S'}(f\wedge\bar{f})/K=0$. If $G(M)$ is transitive on M, then B is satisfied and therefore M is complete. Other examples of manifolds satisfying B are given by the bounded generalized analytic polyhedrons. They can be defined as follows: Let E be a domain in $\mathbf{C}^n$ and let f_q $(q=1,\cdots,k)$ be real analytic functions on E, which can be written in the form $f_q=\sum_{i=1}^\infty f_{qi}\bar{f}_{qi}$, where each f_{qi} is holomorphic in E and the sum is convergent uniformly on every compact subset of E. Let M be a connected component of the set $\{z\in E;\ |f_q(z)|<1 \text{ for } q=1,\cdots,k\}$ such that its closure in $\mathbf{C}^n$ is contained in E and is compact. M is called a generalized bounded analytic polyhedron. The (ordinary) bounded analytic polyhedrons are defined by assuming that the above expression of f_q reduces to one single term: $f_q=f_{q1}\bar{f}_{q1}$.

Another example of a manifold M satisfying condition B is given by a manifold M acted on by a properly discontinuous group D of holomorphic transformations with compact quotient M/D. For these manifolds the following statements are proven: They admit no Killing (nor holomorphic) vector fields commuting with D. The center of any subgroup of $G(M)$ which contains D is discrete. If the scalar curvature R is either $\leqq -n$ or $\geqq -n$ everywhere on M, then $R_{\alpha\beta}=-g_{\alpha\beta}$ and consequently $R=-n$.

The result of H. J. Bremermann is well known [*Lectures on functions of a complex variable*, pp. 349–383, Univ. of Michigan Press, Ann Arbor, 1955; MR **17**, 529; see also F. Sommer and J. Mehring, Math. Ann. **131** (1956), 1–16; MR **17**, 1071], according to which a bounded domain complete with respect to the Bergman metric is a domain of holomorphy. The author remarks that since the kernel function is not intrinsically defined, Bremermann's condition is not intrinsic, while condition B is stronger than Bremermann's but is intrinsic.

E. Vesentini (Pisa)

3018:

Serre, Jean-Pierre. Analogues kählériens de certaines conjectures de Weil. Ann. of Math. (2) **71** (1960), 392–394.

A. Weil [Proc. Internat. Congr. Math. 1954, Amsterdam,

Vol. III, pp. 550–558, Noordhoff, Groningen, 1956; MR **19**, 1078] has given a transcendental proof (valid in the complex case) of the "Castelnuovo formula", concerning the equivalence defect of an algebraic correspondence between two algebraic curves. In this short note the author points out that analogous considerations hold for Kähler manifolds of any dimension. In fact, let V and W be two compact connected Kähler manifolds of the same dimension, and let $H(V)=\sum H^q(V,\mathbf{C})$, $H(W)=\sum H^q(W,\mathbf{C})$ be their cohomology algebras. A correspondence X of V into W is a $\mathbf{C}$-linear application of $H(V)$ into $H(W)$ which is real and keeps the bigradation unchanged. Let X' be the transposed application $H(W)\to H(V)$ with respect to the Poincaré duality. For any integer r, let X_r and X_r' be the restrictions of X and X' to $H^r(V,\mathbf{C})$ and to $H^r(W,\mathbf{C})$. If X is compatible with the Hodge decomposition of $H(V)$ and $H(W)$, then $\mathrm{Tr}(X_r\circ X_r')\geqq 0$. Furthermore, if $\mathrm{Tr}(X_r\circ X_r')=0$, then $X_r=0$.

Let V be an irreducible non-singular projective variety over the field of the complex numbers, and let $f:V\to V$ be a morphism of V into itself. Let us assume that there exists a positive integer p and a hyperplane section E of V such that the divisors $f^{-1}(E)$ and pE are algebraically equivalent. Then for any integer r the proper values of the endomorphism f_r of $H^r(V,\mathbf{C})$, defined by f, have absolute values equal to $p^{r/2}$. *E. Vesentini* (Pisa)

3019:

Vesentini, Edoardo. Osservazioni sulle strutture fibrate analitiche sopra una varietà kähleriana compatta. I. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 232–241.

L'auteur établit, dans cette première partie de son travail [pour la seconde, voir le résumé ci-dessous], certaines conditions suffisantes pour démontrer la trivialité de certains groupes de cohomologie d'une variété kählérienne compacte V à coefficients dans le faisceau des germes de sections holomorphes dans un fibré F de droites complexes multiplicatives. La méthode suivie dans les deux parties est une application particulière d'un théorème de S. Bochner [Canad. J. Math. **3** (1951), 460–470; MR **14**, 90]; plus précisément, c'est une généralisation des résultats analogues obtenus par K. Kodaira [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1268–1273; MR **16**, 618] et Y. Akizuki et S. Nakano [Proc. Japan. Acad. **30** (1954), 266–272; MR **16**, 619]. Les résultats cités ne s'occupant que du cas où la classe caractéristique $c_1(F)$ de F est définie, positive ou négative (c'est-à-dire, représentable par une forme extérieure

$$\gamma = -(2\pi i)^{-1}X_{\alpha\beta^*}dz^\alpha\wedge\overline{dz^\beta}$$

où la matrice $(X_{\alpha\beta^*})$ des coefficients locaux est hermitienne et définie, avec la signature précisée), l'auteur considère le cas plus général, où $c_1(F)$ est semidéfinie. Les calculs explicites sont effectués en employant des systèmes de coordonnées holomorphes géodésiques avec n'importe quel point comme origine, choisis de façon que la matrice $(X_{\alpha\beta^*})$ soit diagonale à l'origine. Les groupes $H^{p,q}(V,F)$ étant représentés biunivoquement par des formes harmoniques de bidegré (p,q) à coefficients dans le faisceau des germes de sections différentiables de F, on a les résultats qui suivent.

Théorème 2.1: Si la classe caractéristique $c_1(F)$ peut être représentée par une forme de courbure γ semidéfinie positive, pour qu'une (p,q)-forme φ dans F avec p et $q>0$ et p ou $q=n=\dim_{\mathbf{C}}(V)$ soit harmonique, il est nécessaire qu'elle satisfasse à l'équation $\gamma\wedge*\varphi=0$.

Théorème 3.1: Si la forme γ est partout semidéfinie positive de rang 1, une (p,q)-forme harmonique φ (avec coefficients dans F) telle que $\gamma\wedge\varphi=0$ identiquement dans V doit satisfaire aussi à l'équation $\gamma\wedge*\varphi=0$.

Par dualité, on obtient des résultats correspondants dans le cas où γ est semidéfinie négative, à savoir les deux théorèmes suivants.

Théorème 4.1: Si la forme γ est semidéfinie négative, une forme harmonique φ à coefficients dans F de bidegré $(0,p)$ ou $(p,0)$, $0\leqq p<n$, doit satisfaire identiquement à $\gamma\wedge\varphi=0$.

Théorème 4.2: Si la forme γ est partout semidéfinie négative et de rang 1, une (p,q)-forme harmonique φ à coefficients dans F, satisfaisant identiquement à $\gamma\wedge*\varphi=0$, doit satisfaire aussi à $\gamma\wedge\varphi=0$.

Dans la suite on suppose que la classe caractéristique $c_1(F)$ de F est représentée par une forme de courbure γ, fermée, réelle, de bidegré (1, 1), semidéfinie négative et partout de rang $r\leqq n=\dim_{\mathbf{C}}(V)$, sauf dans un sous-espace holomorphe compact P de dimension inférieure à n, où le rang de γ peut se réduire éventuellement à des valeurs plus petites que r.

Théorème 5.1: Il n'y a dans V aucune forme harmonique non triviale φ, à coefficients dans F, de bidegré $(p,0)$ ou $(0,p)$ pour $0\leqq p<r$.

Si M est une sous-variété analytique complexe, compacte, immergée régulièrement dans V, de dimension complexe m et non contenue dans la variété singulière P, on a aussi le résultat suivant.

Théorème 5.2: Si la restriction de γ à l'espace tangent de M a le rang partout $<r$ dans M, alors la restriction à l'espace tangent de M de n'importe quelle forme harmonique φ dans V, de bidegré $(p,0)$ ou $(0,p)$, à coefficients dans F, s'annulle identiquement.

Ce dernier théorème a l'application suivante. On envisage une application holomorphe $\pi:V\to B$ sur une variété kählérienne compacte B de dimension complexe $b<n$, qui est partout surjective (c'est-à-dire, avec différentielle de rang b), sauf dans l'image réciproque $\pi^{-1}(L)$ d'une sous-variété holomorphe et compacte (admettant des singularités) $L\subset B$. On se donne un fibré $\tilde F$ holomorphe de droites multiplicatives sur B et l'on considère le fibré $F=\pi^*(\tilde F)$ induit sur V par π. Si $c_1(\tilde F)$ peut être représentée par une forme de courbure $\tilde\gamma$, partout définie négative dans B, la restriction à $\pi^{-1}(\xi)\subset V$, pour tout $\xi\in B-L$, de n'importe quelle forme harmonique φ dans V à coefficients dans F et de bidegré $(p,0)$ ou $(0,p)$ doit s'annuller identiquement. On conclut aussi que

$$H^{p,0}(V,F) = H^{0,p}(V,F) = 0$$

pour tout p dans l'intervalle $0\leqq p<b$.

E. Calabi (Minneapolis, Minn.)

3020:

Vesentini, Edoardo. Osservazioni sulle strutture fibrate analitiche sopra una varietà kähleriana compatta. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 505–512.

Une partie des résultats énoncés dans la première partie de ce travail [résumé ci-dessus] reste vraie lorsqu'on

remplace les fibrés holomorphes de droites complexes par des fibrés holomorphes W d'espaces vectoriels complexes (à m dimensions) sur une variété kählérienne compacte V à n dimensions complexes. Un tel fibré W étant donné, l'auteur le munit d'une structure unitaire dans chaque fibre et en déduit explicitement les composantes locales $H_{j\lambda\alpha;\mu\beta^*}$ de la forme de courbure dans un ouvert $U_j \subset V$ (ici les indices λ, μ se rattachent à une base locale de la structure de fibré vectoriel sur U_j et les indices α, β^* se rapportent à la base naturelle de l'espace tangent de V donnée par un système de coordonnées holomorphes dans U_j). La forme sesquilinéaire définie localement par $H_{j\lambda\alpha;\mu\beta^*}u^{\lambda\alpha}\overline{v^{\mu\beta}}$ détermine d'une façon naturelle une forme hermitienne sur chaque fibre du fibré vectoriel complexe $W \otimes T_V$ (T_V = fibré tangent complexe de V), qu'on peut désigner par $H_{1,0}$; cette forme a une extension, définie grâce à la structure kählérienne de V, à une forme hermitienne $H_{p,q}$ sur les (p, q)-formes extérieures à coefficients dans W. Une application de l'inégalité de S. Nakano [J. Math. Soc. Japan **7** (1955), 1–12; MR **17**, 409] donne le résultat suivant (Lemme 8.1). Si la forme hermitienne $H_{p,q}$ est partout semidéfinie positive pour n'importe quelles valeurs de p, q, pourvu que $\max(p, q) = n$, alors chaque forme harmonique φ dans V de bidegré (p, q) à coefficients dans W satisfait à $H_{p,q}[\varphi] = 0$.

Théorème 9.1: On suppose qu'il y a partout dans V des systèmes locaux de coordonnées holomorphes $z^1, \cdots, z^n$, pour lesquels les composantes $H_{j\lambda\alpha;\mu\beta^*}$ de la forme $H_{1,0}$ s'annullent pour $b < \max(\alpha, \beta) \leqq n$ (b étant donné), tandis que pour $1 \leqq \max(\alpha, \beta) \leqq b$ la forme ainsi réduite est définie positive; alors chaque forme harmonique $\varphi \not\equiv 0$ de type (p, n) [resp. (n, p)] à coefficients dans W satisfait à l'inégalité $dz^1 \wedge \cdots \wedge dz^b \wedge \varphi \not\equiv 0$ [resp. $\varphi \wedge \overline{(dz^1 \wedge \cdots \wedge dz^b)} \not\equiv 0$]. Par conséquent $H^{p,n}(V, W) = H^{n,p}(V, W) = 0$ lorsque $p > b$.

En employant les mêmes calculs pour le fibré vectoriel W^*, on obtient par la théorème de dualité de Serre des résultats analogues dans le cas où la forme $H_{1,0}$ est semidéfinie négative (théorème 10.1 et cor. 10.2). Les hypothèses de ce dernier thèoréme se présentent dans l'exemple intéressant qui suit. Soit π une application holomorphe et presque partout surjective de V sur une variété kählérienne compacte B à b dimensions complexes. Soit $\hat{W}$ un fibré holomorphe sur B d'espaces vectoriels complexes avec structure unitaire tel que la forme de courbure $\hat{H}_{1,0}$ de $\hat{W}$ est partout définie négative dans B. Alors, en regardant le fibré induit $W = \pi^*\hat{W}$ sur V, on déduit que $H^{p,0}(V, W) = H^{0,p}(V, W) = 0$ pour $0 \leqq p < b$; en plus, pour tout p, la restriction de n'importe quelle forme harmonique φ, de bidegré $(p, 0)$ ou $(0, p)$ à coefficients dans W, à l'image réciproque non singulière $\pi^{-1}(\xi)$ de n'importe quel point $\xi \in B$ (sauf les valeurs critique de π) s'annulle identiquement.

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AUTHOR INDEX

PART A